Optimal Morse Flows on non-orientable 3-manifolds

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A flow is called a *Morse flow* if the following conditions hold true:
1) it has finitely many singular points which are non-degenerate;
2) stable and unstable manifolds with singular points have a transversal intersection;
3) $\alpha-$ and $\omega-$limit sets of each trajectory are singular points.

Let M be a smooth closed non-orientable 3-manifold. *A Morse flow diagram* takes the form of a surface and two sets of circles embedded in it. The surface F is the boundary of the regular neighborhood of sources and stable manifolds with singular points of index 1.

On the surface F, we have the following sets of circles:
1) circles u, which are intersections of unstable manifolds of singular points of index 1 with F;
2) circles v, which are intersections of stable manifolds of singular points of index 2 with F.

Morse flow diagram on a three-dimensional manifold is the 3-tuple (F, u, v) consisting of a surface, a set of circles. Two Morse flow diagrams are said to be *equivalent* if there is a surface homeomorphism that maps the sets of arcs and circles into sets of arcs and circles of the same type and preserves framing. According to A.Prishlyak, two Morse flows on 3-manifolds are topologically equivalent if and only if their diagrams are equivalent.

A Morse flow is called *optimal* if it has the minimum number of singular points and trajectories between the saddles. An optimal (polar) Morse flow diagram on a closed 3-manifold is similar to a Heegaard diagram.

We show that if the Heegaard complexity of a non-orientable 3-manifold M is no more than 5, then M is homeomorphic to $L_{p,q}\#S^1\widetilde{\times}S^2$. There exists a Heegaard diagram of complexity 6 of a closed non-orientable 3-manifold which is not homeomorphic to $L_{p,q}\#S^1\widetilde{\times}S^2$.

We show that up to topological equivalence, there exists a unique optimal Morse flow on $S^1\widetilde{\times}S^2$, 2 flows on $L_{2,1}\#S^1\widetilde{\times}S^2$, 2 flows on $L_{3,1}\#S^1\widetilde{\times}S^2$, 3 flows on $L_{4,1}\#S^1\widetilde{\times}S^2$, 3 flows on $L_{5,1}\#S^1\widetilde{\times}S^2$ and 3 flows on $L_{5,2}\#S^1\widetilde{\times}S^2$. There are no other optimal flows on closed non-orientable 3-manifolds of complexity less than 6.

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