On the monoid of cofinite partial isometries of a finite power of positive integers with the usual metric

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We follow the terminology of [1, 2]. For any positive integer n by $\mathcal{S}_n$ we denote the group of permutations of the set $\{1,\ldots,n\}$.

A partial transformation $\alpha\colon (X,d) \rightharpoonup (X,d)$ of a metric space (X,d) is called *isometric* or a *partial isometry*, if $d(x\alpha,y\alpha)=d(x,y)$ for all $x,y\in\operatorname{dom}\alpha$.

For an arbitrary positive integer $n\geqslant 2$ by $\mathbb{N}^n$ we denote the n-th power of the set of positive inters $\mathbb{N}$ with the usual metric:

$$d((x_1,\cdots,x_n),(y_1,\ldots,y_2))=\sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}.$$

Let $\mathbf{I}\mathbb{N}_{\infty}^{n}$ be the set of all partial cofinite isometries of $\mathbb{N}^n$. It is obvious that $\mathbf{I}\mathbb{N}_{\infty}^{n}$ with the operation of composition of partial isometries is an inverse submonoid of the symmetric inverse monoid $\mathcal{I}_{\mathbb{N}}$ over $\mathbb{N}$ and later by $\mathbf{I}\mathbb{N}_{\infty}^{n}$ we shall denote the *monoid of all partial cofinite isometries of* $\mathbb{N}^n$.

Theorem 1. *For any positive integer $n\geqslant 2$ the group of units $H(\mathbb{I})$ of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{n}$ is isomorphic to the symmetric group $\mathcal{S}_n$. Moreover, every element of $H(\mathbb{I})$ is induced by a permutation of the set $\{1,\ldots,n\}$.*

Lemma 2. *Let n be any positive integer $\geqslant 2$. Let α be an arbitrary element of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{n}$. Then there exists the unique element σ_α of the group of units $H(\mathbb{I})$ and the unique idempotents $\varepsilon_{l(\alpha)}$ and $\varepsilon_{r(\alpha)}$ of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}$ such that $\alpha=\sigma_\alpha\varepsilon_{l(\alpha)}=\varepsilon_{r(\alpha)}\sigma_\alpha$.*

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s\preccurlyeq t$ if and only if there exists $e\in E(S)$ such that $s=te$. This order is called the *natural partial order* on S [4].

Theorem 3. *Let n be any positive integer $\geqslant 2$. Let α and β be elements of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}$. Let $\alpha=\sigma_\alpha\varepsilon_{l(\alpha)}=\varepsilon_{r(\alpha)}\sigma_\alpha$ and $\beta=\sigma_\beta\varepsilon_{l(\beta)}=\varepsilon_{r(\beta)}\sigma_\beta$ for some elements σ_α and σ_β of the group of units $H(\mathbb{I})$ and idempotents $\varepsilon_{l(\alpha)}$, $\varepsilon_{r(\alpha)}$, $\varepsilon_{l(\beta)}$ and $\varepsilon_{r(\beta)}$ of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}$. Then $\alpha\preccurlyeq\beta$ in $\mathbf{I}\mathbb{N}_{\infty}^{n}$ if and only if $\sigma_\alpha=\sigma_\beta$, $\varepsilon_{l(\alpha)}\preccurlyeq\varepsilon_{l(\beta)}$ and $\varepsilon_{r(\alpha)}\preccurlyeq\varepsilon_{r(\beta)}$ in $E(\mathbf{I}\mathbb{N}_{\infty}^{n})$.*

A congruence $\mathfrak{C}$ on a semigroup S is called a *group congruence* if the quotient semigroup $S/\mathfrak{C}$ is a group. If $\mathfrak{C}$ is a congruence on a semigroup S then by $\mathfrak{C}^\sharp$ we denote the natural homomorphism from S onto the quotient semigroup $S/\mathfrak{C}$. Every inverse semigroup S admits a *least* (*minimum*) *group congruence* $\mathfrak{C}_{\mathbf{mg}}$:

$$a\mathfrak{C}_{\mathbf{mg}}b \text{ if and only if there exists } e\in E(S) \text{ such that } ae=be$$

(see [3, Lemma III.5.2]).

Theorem 4. *Let n be any positive integer $\geqslant 2$. Then the quotient semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the group $\mathcal{S}_n$ and the natural homomorphism $\mathfrak{C}_{\mathbf{mg}}^\sharp\colon \mathbf{I}\mathbb{N}_{\infty}^{n}\to\mathbf{I}\mathbb{N}_{\infty}^{n}/\mathfrak{C}_{\mathbf{mg}}$ is defined in the following way: $\alpha\mapsto\sigma_\alpha$.*

The following theorem gives the description of Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ on $\mathbf{I}\mathbb{N}_{\infty}^{n}$.

Theorem 5. *Let n be any positive integer $\geqslant 2$ and $\alpha, \beta \in \mathbf{I}\mathbb{N}_{\infty}^{n}$. Then the following statements hold:*

(i) *$\alpha \mathcal{L} \beta$ if and only if there exists an element σ of the group of units $H(\mathbb{I})$ of $\mathbf{I}\mathbb{N}_{\infty}^{n}$ such that $\alpha = \sigma\beta$;*

(ii) *$\alpha \mathcal{R} \beta$ if and only if there exists an element σ of the group of units $H(\mathbb{I})$ of $\mathbf{I}\mathbb{N}_{\infty}^{n}$ such that $\alpha = \beta\sigma$;*

(iii) *$\alpha \mathcal{H} \beta$ if and only if there exist elements σ_1 and σ_2 of the group of units $H(\mathbb{I})$ of $\mathbf{I}\mathbb{N}_{\infty}^{n}$ such that $\alpha = \sigma_1\beta$ and $\alpha = \beta\sigma_2$;*

(iv) *$\alpha \mathcal{D} \beta$ if and only if there exist elements σ_1 and σ_2 of the group of units $H(\mathbb{I})$ of $\mathbf{I}\mathbb{N}_{\infty}^{n}$ such that $\alpha = \sigma_1\beta\sigma_2$;*

(v) *$\mathcal{D} = \mathcal{J}$ on $\mathbf{I}\mathbb{N}_{\infty}^{n}$;*

(vi) *every $\mathcal{J}$-class in $\mathbf{I}\mathbb{N}_{\infty}^{n}$ is finite and consists of incomparable elements with the respect to the natural partial order $\preccurlyeq$ on $\mathbf{I}\mathbb{N}_{\infty}^{n}$.*

Corollary 6. *$\mathbf{I}\mathbb{N}_{\infty}^{n}$ is an E-unitary F-inverse semigroup for any positive integer $n \geqslant 2$.*

The following theorem describes the structure of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}$.

Theorem 7. *Let n be any positive integer $\geqslant 2$. Then the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}$ is isomorphic to the semidirect product $\mathcal{S}_n \ltimes_{\mathfrak{h}} (\mathcal{P}_{\infty}(\mathbb{N}^n), \cup)$ of free semilattice with the unit $(\mathcal{P}_{\infty}(\mathbb{N}^n), \cup)$ by the symmetric group $\mathcal{S}_n$.*

References

[1] A. H. Clifford and G. B. Preston. *The Algebraic Theory of Semigroups*, Vol. I., Amer. Math. Soc. Surveys 7, Providence, R.I., 1961; Vol. II., Amer. Math. Soc. Surveys 7, Providence, R.I., 1967.

[2] M. V. Lawson, *Inverse Semigroups. The Theory of Partial Symmetries*, Singapore: World Scientific, 1998.

[3] M. Petrich, *Inverse Semigroups*, New York: John Wiley & Sons, 1984.

[4] V. V. Wagner, *Generalized groups*, Dokl. Akad. Nauk SSSR, **84**(6): 1119–1122, 1952 (in Russian).