Matrix manifolds as affine varieties

Marek Golasiński
(Faculty of Mathematics and Computer Science, University of Warmia and Mazury, Sloneczna 54 Street, 10-710 Olsztyn, Poland)
E-mail: marekg@matman.uwm.edu.pl

Francisco Gómez Ruiz
(Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias, Universidad de Málaga, Campus Universitario de Teatinos, 29071 Málaga, España)
E-mail: gomez_ruiz@uma.es

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions and write $M_{n,r}(\mathbb{K})$ (resp. $M_n(\mathbb{K})$) for the set of $n \times r$ (resp. $n \times n$) -matrices over $\mathbb{K}$. We aim to examine the structure of special matrix manifolds, Grassmann $G_{n,r}(\mathbb{K})$ and Stiefel manifolds $V_{n,r}(\mathbb{K})$, via their presentations as algebraic sets.

Proposition 1. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Then:*
(1) $G_{n,r}(\mathbb{K}) = \{A \in M_n(\mathbb{K});\ A^2 = A,\ \bar{A}^t = A,\ \mathrm{tr}(A) = r\}$.
(2) $V_{n,r}(\mathbb{K}) = \{A \in M_{n,r}(\mathbb{K});\ \bar{A}^t A = I_r\}$.

This implies that $G_{n,r}(\mathbb{K})$ and $V_{n,r}(\mathbb{K})$ are $\mathbb{R}$-affine varieties. The main result is:

Theorem 2. (1) *The tangent bundle* $TG_{n,r}(\mathbb{K}) = \{(A,B) \in G_{n,r}(\mathbb{K}) \times M_n(\mathbb{K});\ \bar{B}^t = B,\ AB + BA = B\}$*;*
(2) *There is an algebraic isomorphism* $TG_{n,r}(\mathbb{K}) \approx \mathrm{Idem}_{r,n}(\mathbb{K})$, *where* $\mathrm{Idem}_{r,n}(\mathbb{K}) = \{A \in M_n(\mathbb{K});\ A^2 = A,\ \mathrm{rk}(A) = r\}$*;*
(3) *The $\mathbb{C}$-Zariski closure* $\overline{G_{n,r}(\mathbb{C})} = TG_{n,r}(\mathbb{C})$ *for* $G_{n,r}(\mathbb{C})$ *as an $\mathbb{R}$-affine variety.*

Remark 3. The Stiefel map $V_{n,r}(\mathbb{K}) \to G_{n,r}(\mathbb{K})$ given by $A \mapsto A\bar{A}^t$ for $A \in V_{n,r}(\mathbb{K})$ lead to a purely algebraic construction of the universal map

$$EU(\mathbb{K}) \longrightarrow BU(\mathbb{K}).$$

Remark 4. An extension of those results on some matrix manifolds over $\mathbb{K}$ being the Cayley algebra, and more generally, a composition algebra is planed as well.