Lyusternik–Schnirelmann Theorem for C^1-functions on Fréchet spaces

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The Lusternik-Schnirelmann category $\mathrm{Cat}_X A$ of a subset A of a topological space X is the minimal number of closed sets that cover A and each of which is contractible to a point in X. If $\mathrm{Cat}_X A$ is not finite, we write $\mathrm{Cat}_X A = \infty$.

Let $(E, \|\cdot\|^n)$ be a Fréchet space, $\varphi : E \to R$ a C^1-function. Suppose $\mathrm{Cr}(\varphi)$ is the set of critical points of φ and for all $c \in \mathbb{R}$

$$\mathrm{Cr}(\varphi, c) = \{x \in E : \varphi'(x) = 0, \varphi(x) = c\},$$
$$E^c = \{x \in E : \varphi(X) \leqq c\}.$$

Let Co(E) be the set of compact subsets of E. Define the sets

$$\mathcal{A}_i = \big\{A \subset E : A \in \mathrm{Co(E)}, \mathrm{Cat}_E A \geqq i, i \in \mathbb{N}\}, \tag{1}$$

and the numbers

$$\mu_i = \inf_{A \in \mathcal{A}_i} \sup_{x \in A} \varphi(x). \tag{2}$$

Assume $e \in E$, we define

$$\Xi\varphi(e) = \inf \Big\{ \mathrm{d}\,\varphi(e, h) : h \in E, \| \, h \, \|^n = 1, \forall n \in \mathbb{N} \Big\}, \tag{3}$$

where d is the derivative of φ at e in the direction of h.

Definition 1. Let $\varphi : E \to \mathbb{R}$ be a C^1-functional, we say that φ satisfies the Palais-Smale condition at the level c if any sequence $x_i \in E$ such that $\varphi(x_i) \to c$ and $\Xi\varphi(x_i) \to 0$ has the convergent sub-sequence.

Theorem 2. *Suppose E is a Fréchet space and a C^1 function $\varphi : E \to \mathbb{R}$ is bounded below. Suppose $\mathcal{A}_k \neq \emptyset$ for some $k \geqq 1$. If φ satisfies the Palais-Smale conditions at all levels $b = \mu_i, i = 1, \cdots k$, and E^c is complete for each $c \in \mathbb{R}$, then φ has at least k distinct critical points.*