Structure of commutant and centralizer, minimal generating sets of Sylow 2-subgroups $Syl_2 A_n$ of alternating and symmetric groups

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Let $Syl_2 A_{2^k}$ and $Syl_2 A_n$ be Sylow 2-subgroups of corresponding alternating groups A_{2^k} and A_n. We find a least generating set and a structure for such subgroups $Syl_2 A_{2^k}$ and $Syl_2 A_n$. The aim of this investigation is to research the structure of a commutant and a centralizer of $Syl_2 A_n$ and $Syl_2 S_n$ and find numbers of minimal generating sets for $Syl_2 S_{2^k}$ and $Syl_2 A_n$. Let us denote by $X^{[k]}$ a regular truncated binary rooted tree with number of levels from 0 to k, where $X = \{0, 1\}$. The set $X^n \subset X^*$ is called the n-th level of the tree X^* and $X^0 = \{v_0\}$. For every automorphism $g \in AutX^*$ and every word $v \in X^*$ define the section (state) $g_{(v)} \in AutX^*$ of g at v by the rule: $g_{(v)}(x) = y$ for $x, y \in X^*$ if and only if $g(vx) = g(v)y$. The restriction of the action of an automorphism $g \in AutX^*$ to the subtree $X^{[l]}$ is denoted by $g_{(v)}|_{X^{[l]}}$. A restriction $g_{(v)}|_{X^{[1]}}$ is called the vertex permutation (v.p.) of g in a vertex v. The vertex of X^j having the number i we denote by $v_{j,i}$ also we denote by $v_{j,i}X^{[k-j]}$ the subtree of $X^{[k]}$ with a root in $v_{j,i}$. Let β belongs to full automorphism group $AutX^{[k]}$ of $X^{[k]}$.

Definition 1. Let us call the index of an automorphism β on X^l a number of no trivial v.p. of β on X^l.

Definition 2. Define a element of type T as an automorphism $\tau_{i_0,\ldots,i_{2^{k-1}};j_{2^{k-1}},\ldots,j_{2^k}}$, that has even index at X^{k-1} and it has exactly m active states, $m < 2^{k-2}$, in vertexes $v_{k-1,j}$ with number $1 \leq j \leq 2^{k-2}$ and m active states in vertices $v_{k-1,j}$, $2^{k-2} < j \leq 2^{k-1}$. Set of such elements is denoted by T.

Let $n = 2^{k_0} + 2^{k_1} + \ldots + 2^{k_m}$, where $0 \leq k_0 < k_1 < \ldots < k_m$ and $m \geq 0$. Also recall that $Syl_2 S_n = Syl_2 S_{2^{k_0}} \times \ldots \times Syl_2 S_{2^{k_m}}$.

Theorem 3. ***A maximal 2-subgroup*** *of $AutX^{[k]}$ acting by even permutations on X^k has the structure of the semidirect product $G_k \simeq \wr_{i=1}^{k-1} C_2 \ltimes C_2^{2^{k-1}-1}$ and isomorphic to $Syl_2 A_{2^k}$.*

An automorphisms group of the subgroup $C_2^{2^{k-1}-1}$ is based on permutations of copies of C_2. Orders of $\wr_{i=1}^{k-1} C_2$ and $C_2^{2^{k-1}-1}$ are equals. A homomorphism from $\wr_{i=1}^{k-1} C_2$ into $Aut(C_2^{2^{k-1}-1})$ is injective because a kernel of action $\wr_{i=1}^{k-1} C_2$ on $C_2^{2^{k-1}-1}$ is trivial. The group G_k is a proper subgroup of index 2 in the group $\wr_{i=1}^{k} C_2$ [6].

Theorem 4. *The centralizer of $Syl_2 S_{2^{k_i}}$ with $k_i > 2$, in $Syl_2 S_n$ is isomorphic to ${}^{Syl_2 Syl_2 S_n}/_{Syl_2 S_{2^{k_i}}} \times Z(Syl_2 S_{2^{k_i}})$.*

Theorem 5. *The centralizer of $Syl_2 A_{2^{k_i}}$ with $k_i > 2$, in $Syl_2 A_n$ is isomorphic to ${}^{Syl_2 A_n}/_{Syl_2 A_{2^{k_i}}} \times Z(Syl_2 A_{2^{k_i}})$.*

We will call ***diagonal base*** [1], [2] (S_d) for $Syl_2 S_{2^k} \simeq AutX^{[k]}$ a generating set where every j-th generator has odd number of non trivial v.p. on X^j and it has only trivial v.p. on other levels. A number of such tuple with odd number of no trivial v.p. that can be on X^j equals to $2^{2^j - 1}$.

There is minimum one permutation of type T [1] in S_d for $Syl_2A_{2^k}$. It can be chosen in $(2^{n-2})^2$ ways. Thus, general cardinality of S_d for $Syl_2A_{2^k}$ is $2^{2^{k-1}-k-2}(2^{n-2})^2$. Hence, it can be applied [3]. As a result, we have $2^{k(2^k-k-1)} \cdot (2^k-1)(2^k-2)(2^k-2^2)...(2^k-2^{k-1})$ bases for $Syl_2S_{2^k}$.

Lemma 6. *Commutators of all elements from $Syl_2A_{2^k}$ have all possible even indexes on X^l, $l<k-1$ of $X^{[k]}$ and on X^{k-2} of subtrees $v_{11}X^{[k-1]}$ and $v_{12}X^{[k-1]}$.*

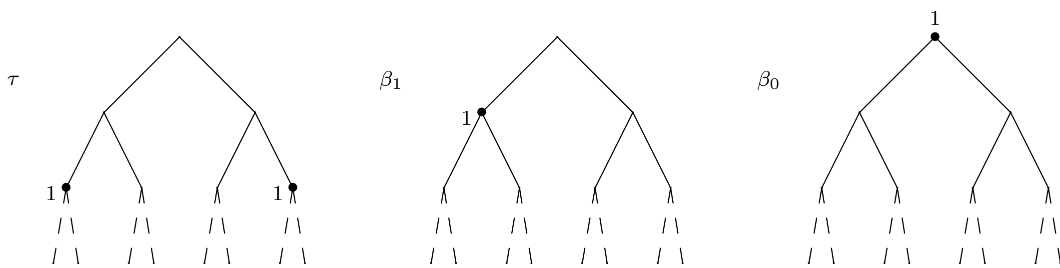
Theorem 7. *The set of all commutators K of Sylow 2-subgroup $Syl_2A_{2^k}$ of the alternating group A_n is the commutant of $Syl_2A_{2^k}$.*

Proposition 8. *Frattini subgroup $\phi(G_k)$ acts by all even permutations on X^l, $1 \le l \le k-1$ and any element of $\phi(G_k)$ has arbitrary even indexes on X^{k-2} of subtrees $v_{11}X^{[k-1]}$ and $v_{12}X^{[k-1]}$* [1]. *Also $\phi(G_k)=(G_k)'$.*

Lemma 9. *A quotient group ${}^{G_k}/_{G_k^2G_k'}$ is isomorphic to $\underbrace{C_2 \times C_2 \times ... \times C_2}_{k}$.*

Theorem 10. *A minimal generating set for a group $Syl_2A_{2^k}$ consists of k elements.*

Example 11. For example, a minimal set of generators for $Syl_2(A_8)$ may be constructed by following way, for convenience let us consider the next set:



Let $n=2^{k_0}+2^{k_1}+...+2^{k_m}$, where $0 \le k_0 < k_1 < ... < k_m$ and $m \ge 0$.

Theorem 12. *Any minimal set of generators for Syl_2A_n has $\sum\limits_{i=0}^{m} k_i - 1$ generators, if $m>0$, and it has k_0 generators, if $m=0$. For instance if $n=4k+2$, then Syl_2A_n has $k_1+k_2+\ldots+k_m$ generators.*

Example 13. Minimal generating set for Syl_2A_{28} has 8 elements: $(25,27)(26,28), (23,24)(25,26),$
$(17,21)(18,22)(19,23)(20,24), (17,19)(18,20), (15,16)(17,18),$
$(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16), (1,5)(2,6)(3,7)(4,8), (1,3)(2,4).$

Rerefences

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