Structure of commutant and centralizer, minimal generating sets of Sylow 2-subgroups $Syl_2 A_n$ of alternating and symmetric groups

Ruslan Skuratovskii
(Kiev, MAUP)
E-mail: ruslcomp@mail.ru

Let $Syl_2 A_{2k}$ and $Syl_2 A_n$ be Sylow 2-subgroups of corresponding alternating groups $A_{2k}$ and $A_n$. We find a least generating set and a structure for such subgroups $Syl_2 A_{2k}$ and $Syl_2 A_n$. The aim of this investigation is to research the structure of a commutant and a centralizer of $Syl_2 A_n$ and $Syl_2 S_n$ and find numbers of minimal generating sets for $Syl_2 S_{2k}$ and $Syl_2 A_n$. Let us denote by $X^{[k]}$ a regular truncated binary rooted tree with number of levels from 0 to $k$, where $X = \{0, 1\}$. The set $X^n \subset X^*$ is called the $n$-th level of the tree $X^*$ and $X^0 = \{v_0\}$. For every automorphism $g \in Aut X^*$ and every word $v \in X^*$ define the section (state) $g(v) \in Aut X^*$ of $g$ at $v$ by the rule: $g(v)(x) = y$ for $x, y \in X^*$ if and only if $g(vx) = g(v)y$. The restriction of the action of an automorphism $g \in Aut X^*$ to the subtree $X^[[i]]$ is denoted by $g(v)|_{X[[i]]}$. A restriction $g(v)|_{X[[i]]}$ is called the vertex permutation (v.p.) of $g$ in a vertex $v$. The vertex of $X^j$ having the number $i$ we denote by $v_{j,i}$ also we denote by $v_{j,i}X^{[k-j]}$ the subtree of $X^{[k]}$ with a root in $v_{j,i}$. Let $\beta$ belongs to full automorphism group $Aut X^{[k]}$ of $X^{[k]}$.

**Definition 1.** Let us call the index of an automorphism $\beta$ on $X^l$ a number of no trivial v.p. of $\beta$ on $X^l$.

**Definition 2.** Define a element of type $T$ as an automorphism $\tau_{i_0, ..., i_{k-1}:j_{k-1}, ..., j_0}$, that has even index at $X^{k-1}$ and it has exactly $m$ active states, $m < 2^{k-2}$, in vertexes $v_{k-1,j}$ with number $1 \leq j < 2^{k-2}$ and $m$ active states in vertexes $v_{k,j}$, $2^{k-2} < j < 2^{k-1}$. Set of such elements is denoted by $T$.

Let $n = 2^{k_0} + 2^{k_1} + ... + 2^{k_m}$, where $0 \leq k_0 < k_1 < ... < k_m$ and $m \geq 0$. Also recall that $Syl_2 S_n = Syl_2 S_{2^{k_0}} \times ... \times Syl_2 S_{2^{k_m}}$.

**Theorem 3.** A maximal 2-subgroup of $Aut X^{[k]}$ acting by even permutations on $X^k$ has the structure of the semidirect product $G_k \simeq \prod_{i=1}^{k-1} C_2 \ltimes C_2^{k-1-i}$ and isomorphic to $Syl_2 A_{2k}$.

An automorphisms group of the subgroup $C_2^{k-1-i}$ is based on permutations of copies of $C_2$. Orders of $\prod_{i=1}^{k-1} C_2$ and $C_2^{k-1-i}$ are equals. A homomorphism from $\prod_{i=1}^{k-1} C_2$ into $Aut(C_2^{k-1-i})$ is injective because a kernel of action $\prod_{i=1}^{k-1} C_2$ on $C_2^{k-1-i}$ is trivial. The group $G_k$ is a proper subgroup of index 2 in the group $\prod_{i=1}^{k} C_2$.

**Theorem 4.** The centralizer of $Syl_2 S_{2k_i}$ with $k_i > 2$, in $Syl_2 S_n$ is isomorphic to $Syl_2 S_{2k_i} / Syl_2 S_{2k_i} \times Z(Syl_2 S_{2k_i})$.

**Theorem 5.** The centralizer of $Syl_2 A_{2k_i}$ with $k_i > 2$, in $Syl_2 A_n$ is isomorphic to $Syl_2 A_n / Syl_2 A_{2k_i} \times Z(Syl_2 A_{2k_i})$.

We will call diagonal base [1, 2] $(S_{ij})$ for $Syl_2 S_{2k} \simeq Aut X^{[k]}$ a generating set where every $j$-th generator has odd number of non trivial v.p. on $X^j$ and it has only trivial v.p. on other levels. A number of such tuple with odd number of no trivial v.p. that can be on $X^j$ equals to $2^{j-1}$. 

1
There is minimum one permutation of type $T$ \([1]\) in $S_d$ for $Syl_2 A_{2k}$. It can be chosen in $(2^{n-2})^2$ ways. Thus, general cardinality of $S_d$ for $Syl_2 A_{2k}$ is $2^{k+1-1-k} - 2(2^{n-2})^2$. Hence, it can be applied \([3]\). As a result, we have $2^{k(2^k-k-1)} (2^{k-1})(2^k-2)(2^k-2^2)...(2^k-2^{k-1})$ bases for $Syl_2 A_{2k}$.

**Lemma 6.** Commutators of all elements from $Syl_2 A_{2k}$ have all possible even indexes on $X^l$, $l < k-1$ of $X^k$ and on $X^{k-2}$ of subtrees $v_{11}X^{[k-1]}$ and $v_{12}X^{[k-1]}$.

**Theorem 7.** The set of all commutators $K$ of Sylow 2-subgroup $Syl_2 A_{2k}$ of the alternating group $A_n$ is the commutant of $Syl_2 A_{2k}$.

**Proposition 8.** Frattini subgroup $\phi(G_k)$ acts by all even permutations on $X^l$, $1 \leq l \leq k-1$ and any element of $\phi(G_k)$ has arbitrary even indexes on $X^{k-2}$ of subtrees $v_{11}X^{[k-1]}$ and $v_{12}X^{[k-1]}$ \([1]\). Also $\phi(G_k) = (G_k)'$.

**Lemma 9.** A quotient group $G_k \cong G'_k \times C_2 \times \cdots \times C_2$. $k$

**Theorem 10.** A minimal generating set for a group $Syl_2 A_{2k}$ consists of $k$ elements.

**Example 11.** For example, a minimal set of generators for $Syl_2 (A_8)$ may be constructed by following way, for convenience let us consider the next set:

Let $n = 2^{k_0} + 2^{k_1} + \ldots + 2^{k_m}$, where $0 \leq k_0 < k_1 < \ldots < k_m$ and $m \geq 0$.

**Theorem 12.** Any minimal set of generators for $Syl_2 A_n$ has $\sum_{i=0}^{m} k_i - 1$ generators, if $m > 0$, and it has $k_0$ generators, if $m = 0$. For instance if $n = 4k+2$, then $Syl_2 A_n$ has $k_1$ generators, if $m = 0$.

**Example 13.** Minimal generating set for $Syl_2 A_{28}$ has 8 elements: (25, 27)(26, 28), (23, 24)(25, 26), (17, 21)(18, 22)(19, 23)(20, 24), (17, 19)(18, 20), (15, 16)(17, 18), (1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(8, 16), (1, 5)(2, 6)(3, 7)(4, 8), (1, 3)(2, 4).

**References**


