# On the co-adjont orbits in Grassmann algebras and their applications

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In the classical works [2, 3, 4] still in 1928 the French mathematician M. A. Buhl posed the problem of classifying all infinitesimal symmetries of a given linear vector field equation

$$A\psi = 0, \tag{1}$$

where the function $\psi \in C^2(\mathbb{R}^n;\mathbb{R})$, and

$$A := \sum_{j=\overline{1,n}} a_j(x)\frac{\partial}{\partial x_j} \tag{2}$$

is a vector field operator on $\mathbb{R}^n$ with coefficients $a_j \in C^1(\mathbb{R}^n;\mathbb{R})$, $j=\overline{1,n}$. It is easy to show that the problem under regard is reduced [10] to describing all possible vector fields

$$A^{(k)} := \sum_{j=\overline{1,n}} a_j^{(k)}(x)\frac{\partial}{\partial x_j} \tag{3}$$

with coefficients $a_j^{(k)} \in C^1(\mathbb{R}^n;\mathbb{R}), j,k=\overline{1,n}$, satisfying the Lax type commutator condition

$$[A, A^{(k)}] = 0 \tag{4}$$

for all $x \in \mathbb{R}^n$ and $k=\overline{1,n}$. The M.A. Buhl problem above was completely solved in 1931 by the Ukrainian mathematician G. Pfeiffer in the works [7, 8, 9, 10, 11, 12], where he has constructed explicitly the searched set of independent vector fields (3)), having made use effectively of the full set of invariants for the vector field (2) and the related solution set structure of the Jacobi-Mayer system of equations, naturally following from (4)). Some results, yet not complete, were also obtained by C. Popovici in [14].

Some years ago the M.A. Buhl type equivalent problem was independently reanalyzed once more by Japanese mathematicians K. Takasaki and T. Takebe [18, 19] and later by L. V. Bogdanov, V. S. Dryuma and S. V. Manakov for a very special case when the vector field operator (2) depends analytically on a "spectral" parameter $\lambda \in \mathbb{C}$ :

$$\tilde{A} := \frac{\partial}{\partial t} + \sum_{j=\overline{1,n}} a_j(t,x;\lambda)\frac{\partial}{\partial x_j} + a_0(t,x;\lambda)\frac{\partial}{\partial \lambda}. \tag{5}$$

Based on the before developed Sato theory [16, 17], the authors mentioned above have shown for some special kinds of vector fields (5) that there exists an infinite hierarchy of the symmetry vector fields

$$\tilde{A}^{(k)} := \frac{\partial}{\partial \tau_k} + \sum_{j=\overline{1,n}} a_j^{(k)}(\tau,x;\lambda)\frac{\partial}{\partial x_j} + a_0^{(k)}(\tau,x;\lambda)\frac{\partial}{\partial \lambda}, \tag{6}$$

where $\tau = (t;\tau_1,\tau_2,...) \in \mathbb{R}^{\mathbb{Z}_+}, k \in \mathbb{Z}_+$, satisfying the Lax-Sato type compatible commutator conditions

$$[\tilde{A}, \tilde{A}^{(k)}] = 0 = [\tilde{A}^{(j)}, \tilde{A}^{(k)}] \tag{7}$$

for all $k,j \in \mathbb{Z}_+$. Moreover, in the cases under regard, the compatibility conditions (7) proved to be equivalent to some very important for applications heavenly type dispersionless equations in partial derivatives.

In the present work we investigate the Lax–Sato compatible systems, the related Lie-algebraic structures and complete integrability properties of an interesting class of nonlinear dynamical systems called the heavenly type equations, which were introduced by Plebański [13] and analyzed in a series of articles [18, 19]. In our work, having employed the AKS-algebraic and related $\mathcal{R}$-structure schemes [1, 20], applied to the holomorphic loop Lie algebra $\tilde{\mathcal{G}} := \widetilde{diff}(\mathbb{T}^n)$ of vector fields on torus $\mathbb{T}^n, n \in \mathbb{Z}_+$, the orbits of the corresponding coadjoint actions on $\tilde{\mathcal{G}}^*$, closely related to the classical Lie-Poisson type structures, were reanalyzed and studied in detail. By constructing two commuting flows on the coadjoint space $\tilde{\mathcal{G}}^*$, generated by a chosen root element $\tilde{l} \in \tilde{\mathcal{G}}^*$ and some Casimir invariants, we have successively demonstrated that their compatibility condition coincides exactly with the corresponding heavenly equations under consideration.

As a by-product of the construction, devised in the work [15], we prove that all the heavenly equations have a similar origin and can be represented as a Lax compatibility condition for special loop vector fields on the torus $\mathbb{T}^n$. We analyze the structure of the infinite hierarchy of conservations laws, related to the heavenly equations, and demonstrate their analytical structure connected with the Casimir invariants is generated by the Lie–Poisson structure on $\tilde{\mathcal{G}}^*$. Moreover, we have extended the initial Lie-algebraic structure for the case when the the basic Lie algebra $\tilde{\mathcal{G}} = \widetilde{diff}(\mathbb{T}^n)$ is replaced by the adjacent holomorphic Lie algebra $\bar{\mathcal{G}} := diff_{hol}(\mathbb{C} \times \mathbb{T}^n) \subset diff(\mathbb{C} \times \mathbb{T}^n)$ of vector fields on $\mathbb{C} \times \mathbb{T}^n$. Typical examples are presented for all cases of the heavenly equations and it is shown in detail and their integrability is demonstrated using the scheme devised here. This scheme makes it possible to construct a very natural derivation of well known Lax-Sato representation for an infinite hierarchy of heavenly equations, related to the canonical Lie-Poisson structure on the adjoint space $\bar{\mathcal{G}}^*$. We also briefly discuss the Lagrangian representation of these equations following from their Hamiltonicity with respect to both intimately related commuting evolutionary flows, and the related bi-Hamiltonian structure as well as the Bäcklund transformations. As a matter of fact, there are only a few examples of multi-dimensional integrable systems for which such a detailed description of their mathematical structure has been given. As was already aptly mentioned, the heavenly equations comprise an important class of such integrable systems. This is due in part to the fact that some of them are obtained by a reduction of the Einstein equations with Euclidean (and neutral) signature for (anti-) self-dual gravity, which includes the theory of gravitational instantons. This and other cases of important applications of multi-dimensional integrable equations strongly motivated us to study this class of equations and the related mathematical structures. As a very interesting aspect of our approach to describing integrability of the heavenly dynamical systems, there is a very interesting Lagrange-d'Alembert type mechanical interpretation. We need to underline here that the main motivating idea behind this work was based both on the paper by Kulish, devoted to studying the super-conformal Korteweg-de-Vries equation as an integrable Hamiltonian flow on the adjoint space to the holomorphic loop Lie superalgebra of super-conformal vector fields on the circle, and on the insightful investigation by Mikhalev, which studied Hamiltonian structures on the adjoint space to the holomorphic loop Lie algebra of smooth vector fields on the circle. We were also impressed by deep technical results [18, 19] of Takasaki and Takebe, who fully realized the vector field scheme of the Lax-Sato theory. Additionally, we were strongly influenced both by the works of Konopelchenko [5, 6], as well as by the work of Ferapontov and Moss, in which they devised new effective differential-geometric and analytical methods for studying an integrable degenerate multi-dimensional dispersionless heavenly type hierarchy of equations, the mathematical importance of which is still far from being properly appreciated.

## Rerefences


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