The distributions of random incomplete sums of a series with essential overlaps of cylindrical intervals

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Properties of infinite Bernoulli convolutions are being studied for almost a century. The interest to them due to various reasons has increased considerably in recent years [3]. In the theory of infinite Bernoulli convolutions, there are many complicated problems of probabilistic nature. One of these problems is to extend a theorem due to Jessen and Wintner telling that the distribution of the sum of a random series with independent discrete terms is pure, that is it is either purely discrete, or continuous, or singular. The Jessen–Wintner theorem does not tell us when each of these three cases occur. Another problem is related to the topological, metric, and fractal properties of the spectrum of the distribution, that is to the set of growth of a distribution function.The third one deals with the behavior at infinity of the absolute value of a characteristic function. None of these problems has a solution in the general case but partial solutions are known for several particular cases. One of these particular cases is considered in the current paper, namely we study the random variable

$$\xi = \sum_{n=1}^{\infty} d_n \xi_n, \tag{1}$$

where

$$1) \quad r_0 = \sum_{k=1}^{\infty} d_k = c_1 + \underbrace{c_2 + c_2}_{2} + \underbrace{c_3 + c_3 + c_3 + c_3}_{4} + \ldots + \underbrace{c_n + \ldots + c_n}_{2^{n-1}} + \widetilde{r}_n - \tag{2}$$

is a convergent series with positive terms satisfying the following conditions:

$$\frac{c_n}{\widetilde{r}_n} = n + 1 = \frac{d_m}{r_m}, \quad m = 2^k - 1, \quad k = 1, 2, 3, \ldots\ , \tag{3}$$

$$\widetilde{r}_m = r_{2^n - 1} = \sum_{k=2^n}^{\infty} d_k = \underbrace{c_{n+1} + \ldots + c_{n+1}}_{2^n} + \underbrace{c_{n+2} + \ldots + c_{n+2}}_{2^{n+1}} + \ldots =$$

$$= 2^n c_{n+1} + 2^{n+1} c_{n+2} + 2^{n+2} c_{n+3} + \ldots\ ,$$

$$c_n = d_{2^{n-1}} = d_{2^{n-1}+1} = d_{2^{n-1}+2} = \ldots = d_{2^n - 1}.$$

2) (ξ_n) — is a sequence of independent random variables with the distributions:

$$P\{\xi_n = 0\} = p_{0n} \geq 0, \quad P\{\xi_n = 1\} = p_{1n} \geq 0, \quad p_{0n} + p_{1n} = 1.$$

The properties of the distribution of the random variable ξ are uniquely determined by the sequence of terms (d_n) and by the stochastic matrix $\|p_{in}\|$.

Lemma 1. *The terms and the remainder of a series* (2) *satisfy the following conditions:*

$$c_n = (n+1) \prod_{k=1}^{n} \frac{1}{2^{k-1}(k+1) + 1}, \quad \widetilde{r}_n = \prod_{k=1}^{n} \frac{1}{2^{k-1}(k+1) + 1}.$$

Definition 2. If M is a subset of the set of positive integer numbers $\mathbb{N}$, then $\sum\limits_{n\in M\subset\mathbb{N}} d_n$ is called *a sub-series of series* (2) and its sum $x = x(M)$ is called *an incomplete sum of series* (2). The set of all incomplete sums of series (2) is denoted by $E\{d_n\}$.

Definition 3. [6] *The spectrum* S_ξ of the distribution of the random variable ξ is called the set of points of growth of the distribution function $F_\xi(x)$ (alternatively, S_ξ is called the minimal closed support), that is

$$S_\xi = \{x : F_\xi(x+\varepsilon) - F_\xi(x-\varepsilon) = P\{\xi \in (x-\varepsilon; x+\varepsilon)\} > 0, \forall\varepsilon > 0\}.$$

Lemma 4. *If $p_{in} > 0$ for all $i \in \{0,1\}$ and all $n \in \mathbb{N}$, then the spectrum S_ξ of the distribution of the random variable ξ coincides with the set $E\{d_n\}$ of all incomplete sums of series* (2)*, that is*

$$S_\xi = E\{d_n\} \equiv \left\{x : x = \sum_{n\in\mathbb{N}} d_n, \quad M \in 2^{\mathbb{N}}\right\}.$$

Definition 5. A Lebesgue null-set of the space $\mathbb{R}^1$ whose Hausdorff–Besicovitch dimension equals 1 is called *superfractal*, while a continuum whose Hausdorff–Besicovitch dimension equals zero is called *anomalous fractal.*

Theorem 6. *The set of all incomplete sums of series* (2) *satisfying homogeneity condition* (3) *is a superfractal.*

Corollary 7. *The spectrum S_ξ of the distribution of the random variable ξ is a superfractal set.*

Theorem 8. *In the continuous case, that is if ($L = 0$) the distribution of the random variable ξ is a singular Cantor type distribution with a superfractal spectrum.*

Definition 9. A characteristic function $f_\xi(t)$ of the random variable ξ is called a mathematical expectation of the complex-valued random variable $e^{it\xi}$, that is $f_\xi(t) = \mathbf{M}e^{it\xi}$.

The apparatus of characteristic functions is convenient for studying the structure and properties of distributions of really significant random variables. In particular, it is known [5] that the value $L_\xi = \lim\limits_{|t|\to\infty} \sup |f_\xi(t)|$ is equal

1) 1, if ξ has a discrete distribution;

2) 0, if ξ has an absolutely continuous distribution.

For singular distributions L_ξ can take all values from $[0,1]$. Singular distributions with $L_\xi = 1$ are close to discrete ones, while singular distributions with $L_\xi = 0$ are close to absolutely continuous ones.

Theorem 10. *The following equality holds $L_\xi = \lim\limits_{|t|\to\infty} \sup |f_\xi(t)| = 1$.*

Rerefences

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