Topological invariants of quadratic mappings

Giorgi Khimshiashvili
(3/5, K.Cholokashvili Avenue, Tbilisi, Georgia)
E-mail: giorgi.khimshiashvili@iliauni.edu.ge

Various topological problems concerned with quadratic mappings between real vector spaces play important role in non-linear analysis, optimization theory and theory of integrable systems, to name only a few topics (see, e.g., [1], [2]). In this talk, we present a number of recent results in the spirit of approach suggested in [1], which have been obtained using our results on topological invariants of real polynomial mappings [3], [4]. The leading idea is to elaborate upon certain known results in the natural and practically important context of stable quadratic mappings. For our purposes, it is convenient to use the notion of stability in the sense of singularity theory, i.e., with respect to the right-left (RL) equivalence in C^∞-category. Along these lines, we discuss topological invariants of two classes of quadratic mappings: stable quadratic endomorphisms and proper homogeneous quadratic mappings with fibres of positive dimension. In the latter case we assume that restrictions of such mappings onto the unit sphere S^{n-1} in the source space $\mathbb{R}^n$ are stable in the above sense.

The first main result is concerned with stable quadratic endomorphisms. Recall that in this case the mapping degree $\mathrm{Deg}F$ is defined and its absolute value is a topological invariant [4].

Theorem 1. *The topological degree of stable quadratic endomorphism F of $\mathbb{R}^n$ is equal to the topological degree of its homogeneous part. Moreover,* $\mathrm{Deg}F$ *vanishes for odd n and does not exceed $\frac{(3k-1)!}{k!(2k-1)!}$ for even dimension $n = 2k$.*

Let now $F = (f_1, \dots f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a homogeneous quadratic mapping which is non-degenerate, i.e., has an isolated zero at the origin. In this case, certain topological information can be obtained by considering the origin O_n as an isolated singular point of mapping F. It is well known and easy to verify that components of such a mapping form a system of parameters in the local ring of the origin. In other words, in this case the origin is an isolated singular point of complete intersection. Thus its Milnor number at the origin is defined and one can calculate it using results of [5], which gives our second main result.

Theorem 2. *The Milnor number of non-degenerate homogeneous quadratic mapping*

$$F = (f_1, \dots f_m) : \mathbb{R}^n \to \mathbb{R}^m, \qquad n > m$$

is equal to $\sum_{p=0}^{m-1} (-1)^p \, 2^p \frac{n!}{p!(n-p)!}$.

This result has some immediate topological corollaries.

Corollary 3. *Mapping F is not stable with respect to RL-equivalence.*

Indeed, the value of Milnor number given above is always bigger than $2n$ which according to well known results of J.Mather is the maximal possible value of Milnor number of RL-stable singularity for these dimensions. Having in mind applications to the numerical range mapping of square matrix discussed in the sequel, we present a special case of this result concerned with mappings into the plane.

Corollary 4. *The Milnor number of homogeneous quadratic mapping of $\mathbb{R}^n$ into the plane is equal to* $2n - 1$.

Since a homogeneous quadratic mapping is not stable it is natural to wonder if its restriction to the unit sphere in the source space can be stable in some cases. Simple examples show that this is not always so but it turns out that generically this is the case.

Proposition 5. *For a generic proper homogeneous quadratic mapping, its restriction F_s to the unit sphere is RL-stable.*

In this case, using results of [1], [4] one can obtain an estimate for the Euler characteristics of the projectivized fibres of F defined as the fibres of F_s.

Theorem 6. *The absolute value of the Euler characteristic of projectivized fibres of stable quadratic mapping F does not exceed $\frac{(4n+3)!}{(3n+3)!n!}$.*

As a concrete application of our approach we present some results about numerical range mappings of complex matrices. Let A be a complex $n \times n$ matrix and $W_A : S^{2n-1} \to \mathbb{C}$ be its numerical range mapping. This mapping is a classical object of study in linear algebra. In particular, the famous Hausdorff-Toeplitz theorem states that its image is convex. It is also known that its discriminant consists of $n-1$ closed curves inside the convex domain $\operatorname{im} W$. However not much seems to be known about the topology of its fibres. We clarify the latter issue for $n = 2$ and $n = 3$. Namely, the results presented above combined with results from [2] yield the following conclusion.

Proposition 7. *For $n = 2$, any regular fibre of W_A is diffeomorphic to the circle and any regular fibre of V consists of a single point. Any singular reduced fibre of W_A is homeomorphic to wedge of two circles.*

For $n = 3$, we can describe topology of all regular fibres.

Proposition 8. *For $n = 3$, the regular fibres of W_A are diffeomorphic to the three-torus or the unit tangent bundle of 2-sphere.*

Rerefences

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