Let $G$ and $H$ be two groups acting on path connected topological spaces $X$ and $Y$ respectively. Assume that $H$ is finite of order $m$ and the quotient maps $p : X \to X/G$ and $q : Y \to Y/H$ are regular coverings. Then it is well-known that the wreath product $G \wr H$ naturally acts on $W = X^m \times Y$, so that the quotient map $r : W \to W/(G \wr H)$ is also a regular covering. We give an explicit description of $\pi_1(W/(G \wr H))$ as a certain wreath product $\pi_1(X/G) \wr_{\partial Y} \pi_1(Y/H)$ corresponding to a non-effective action of $\pi_1(Y/H)$ on the set of maps $H \to \pi_1(X/G)$ via the boundary homomorphism $\partial_Y : \pi_1(Y/H) \to H$ of the covering map $q$.

Such a statement is known and usually exploited only when $X$ and $Y$ are contractible, in which case $W$ is also contractible, and thus $W/(G \wr H)$ is the classifying space of $G \wr H$.

The applications are given to the computation of the homotopy types of orbits of typical smooth functions $f$ on orientable compact surfaces $M$ with respect to the natural right action of the groups $\mathcal{D}(M)$ of diffeomorphisms of $M$ on $\mathcal{C}^\infty(M, \mathbb{R})$.

References