Inner harmonic measure for the fractional Laplacian

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The talk is based on [9], and it deals with the theory of potentials with respect to the α-Riesz kernel $|x-y|^{\alpha-n}$ of order $\alpha \in (0,2]$ on $\mathbb{R}^n$, $n \geqslant 3$. We first focus on the inner α-harmonic measure ε_y^A for $A \subset \mathbb{R}^n$ arbitrary, being motivated by the known fact that it is the main tool in solving the generalized Dirichlet problem for α-harmonic functions (see [1, 7]). Here ε_y is the unit Dirac measure at $y \in \mathbb{R}^n$, and μ^A the inner α-Riesz balayage of a Radon measure μ to $A \subset \mathbb{R}^n$ (see [8], cf. also [4] where $\alpha = 2$).

We describe the Euclidean support of the inner α-harmonic measure ε_y^A, provide a formula for evaluation of its total mass $\varepsilon_y^A(\mathbb{R}^n)$, establish the vague continuity of the map $y \mapsto \varepsilon_y^A$ outside the inner α-irregular points for A, and obtain necessary and sufficient conditions for ε_y^A to be of finite energy (more generally, for ε_y^A to be absolutely continuous with respect to inner capacity) as well as for $\varepsilon_y^A(\mathbb{R}^n) \equiv 1$ to hold. Those criteria are given in terms of newly defined concepts of inner α-thinness and inner α-ultrathinness of A at infinity (see [9]) that for $\alpha = 2$ and A Borel coincide with the concepts of outer 2-thinness at infinity by Doob [5] and Brelot [2], respectively.

Further, we extend some of these results to μ^A general by verifying the integral representation formula for inner balayage:

$$\mu^A = \int \varepsilon_y^A \, d\mu(y).$$

We also show that for every $A \subset \mathbb{R}^n$, there exists a K_σ-set $A_0 \subset A$ such that

$$\mu^A = \mu^{A_0} \quad \text{for all } \mu,$$

and give various applications of this theorem. In particular, we prove the vague and strong continuity of the inner swept, resp. inner equilibrium, measure under an approximation of A arbitrary, thereby strengthening Fuglede's result [6], established for A Borel.

Being mainly new even for $\alpha = 2$, the results obtained also present a further development of the theory of inner Newtonian capacities and of inner Newtonian balayage, originated by Cartan [3, 4].

References

[1] J. Bliedtner, W. Hansen. *Potential Theory. An Analytic and Probabilistic Approach to Balayage.* Springer, 1986.
[2] M. Brelot. Sur le rôle du point à l'infini dans la théorie des fonctions harmoniques. *Ann. Éc. Norm. Sup.* 61 : 301–332, 1944.
[3] H. Cartan. Théorie du potentiel newtonien: énergie, capacité, suites de potentiels. *Bull. Soc. Math. Fr.* 73 : 74–106, 1945.
[4] H. Cartan. Théorie générale du balayage en potentiel newtonien. *Ann. Univ. Fourier Grenoble*, 22 : 221–280, 1946.
[5] J.L. Doob. *Classical Potential Theory and Its Probabilistic Counterpart.* Springer, 1984.
[6] B. Fuglede. On the theory of potentials in locally compact spaces. *Acta Math.*, 103 : 139–215, 1960.
[7] N.S. Landkof. *Foundations of Modern Potential Theory.* Springer, 1972.
[8] N. Zorii. A theory of inner Riesz balayage and its applications. *Bull. Pol. Acad. Sci. Math.*, 68 : 41–67, 2020.
[9] N. Zorii. Harmonic measure, equilibrium measure, and thinness at infinity in the theory of Riesz potentials. *Potential Anal.*, DOI 10.1007/s11118-021-09923-2, 2021.