Extremal problem for non-overlapping domains with free poles

Liudmyla Vyhivska
(Kyiv, Ukraine)
E-mail: liudmylavyhivska@ukr.net

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be the Riemann sphere, and $r(B,a)$ be the inner radius of the domain $B \in \overline{\mathbb{C}}$ with respect to the point $a \in B$.

Consider the following problem which was formulated in 1994 [1].

Problem 1. Consider the product

$$I_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where B_0, B_1,...,B_n $(n \geq 2)$ are pairwise non-overlapping domains in $\overline{\mathbb{C}}$ and $a_0 = 0$ and $|a_k| = 1$ for $k = \overline{1,n}$, and $0 < \gamma \leqslant n$. Show that it attains its maximum at a configuration of domains B_k and points a_k possessing rotational n-symmetry.

This problem has a solution only if $\gamma \leqslant n$ as soon as $\gamma = n + \epsilon, \epsilon > 0$, the problem has no solution. Currently it still unsolved in general, only partial results are known [2].

The following theorem holds [3].

Theorem 2. *Let $n \in \mathbb{N}$ and $n \geqslant 2$. Then for any $\beta \in (0; \frac{1}{2}]$ there exists $n_0(\beta)$ such that for all $n \geqslant n_0(\beta)$ and for all $\gamma \in (1, n^{\beta}]$ and for any different points of a unit circle and for any different system of non-overlapping domains B_k, such that $a_k \in B_k \subset \overline{\mathbb{C}}$ for $k = \overline{1,n}$, and $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, the following inequality holds*

$$r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{4}{n}\right)^{n} \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{\gamma}{n^2}\right)^{n + \frac{\gamma}{n}}} \left(\frac{1 - \frac{\sqrt{\gamma}}{n}}{1 + \frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}. \tag{1}$$

Equality is attained if a_k and B_k for $k = \overline{0,n}$, are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2 - \gamma)w^n + \gamma}{w^2(w^n - 1)^2}\, dw^2.$$

References

[1] Vladimir Dubinin. Symmetrization in the geometric theory of functions of a complex variable. *Russian Mathematical Surveys*, 49(1) : 1–79, 1994.

[2] Alexandr Bakhtin, Galina Bakhtina, Yurii Zelinskii. *Topological-algebraic structures and geometric methods in complex analysis*, volume 73 of *In Proceedings of the Institute of Mathematics of NAS of Ukraine*. Kyiv: Institute of Mathematics of NAS of Ukraine, 2008.

[3] Alexandr Bakhtin, Liudmyla Vyhivska. Problem on extremal decomposition of the complex plane with free poles. *Journal of Mathematical Sciences*, 248(2) : 145–165, 2020.