Modular knots obey the Chebotarev law

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A *Chebotarev link* in S^3 is an analogue of the set of all prime numbers in $\mathbb{Z}$. It would play an important roll in arithmetic topology, especially when we formulate an analogue of the idelic class field theory for 3-manifolds [Uek21a] (see also [Mor12, Nii14, NU19, Mih19]). Here is the definition:

Definition 1 (The Chebotarev law)**.** Let $(K_i)_i = (K_i)_{i\in\mathbb{N}_{>0}}$ be a sequence of disjoint knots in a 3-manifold M. For each $n \in \mathbb{N}_{>0}$ and $j > n$, put $L_n = \cup_{i\leq n} K_i$ and let $[K_j]$ denote the conjugacy class of K_j in $\pi_1(M - L_n)$. We say that $(K_i)_i$ *obeys the Chebotarev law* if the density equality

$$\lim_{\nu\to\infty} \frac{\#\{n < j \leq \nu \mid \rho([K_j]) = C\}}{\nu} = \frac{\#C}{\#G}$$

holds for any $n \in \mathbb{N}_{>0}$, any surjective homomorphism $\rho : \pi_1(M - L_n) \to G$ to any finite group, and any conjugacy class $C \subset G$.

In order to answer Mazur's question on the existence of such a link in S^3 [Maz12], by using Parry–Pollicott's zeta functions of symbolic dynamics [PP90], McMullen proved a highly interesting theorem:

Proposition 2 ([McM13, Theorem 1.2])**.** *Let $(K_i)_i$ be the closed orbits of any topologically mixing pseudo-Anosov flow on a closed 3-manifold M, ordered by length in a generic metric. Then $(K_i)_i$ obeys the Chebotarev law.*

Applying his theorem to the monodromy suspension flow of the figure-eight knot K and noting that the Chebotarev law persists under Dehn surgeries, he constructed a Chebotarev link containing K in S^3 [McM13, Corollary 1.3]. We refine his construction in two ways to verify the following assertion:

Theorem 3 ([Uek21b, Theorem 3])**.** *Let L be a fibered hyperbolic link in S^3 and let $(K_i)_i$ denote the sequence of knots consisting of the closed orbits of the suspension flow of the monodromy map and L itself. Then $(K_i)_i$ obeys the Chebotarev law, if ordered by length with respect to a generic metric.*

The union $\mathcal{L} = \cup_i K_i$ is a stably Chebotarev link*, that is, for any finite branched cover $h : M \to S^3$ branched along any finite link in $\mathcal{L}$, the inverse image $h^{-1}(\mathcal{L})$ is again Chebotarev.*

One way is to extend McMullen's theorem for *generalized pseudo-Anosov flows*, which allow 1-pronged singular orbits. The other is to invoke the notion of *rational Fried surgeries*, which produce many (generalized) pseudo-Anosov flows.

Our refinement further provides a new example called modular knots, that are also known as Lorenz knots. Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ denote the upper half plane. The unit tangent bundle of the modular orbifold $\mathrm{PSL}_2\mathbb{Z} \setminus \mathbb{H}^2$ is well-known to be homeomorphic to both the quotient space $\mathrm{PSL}_2\mathbb{Z} \setminus \mathrm{PSL}_2\mathbb{R} \cong \mathrm{SL}_2\mathbb{Z} \setminus \mathrm{SL}_2\mathbb{R}$ and the exterior of a trefoil K in S^3. A flow on $\mathrm{PSL}_2\mathbb{Z} \setminus \mathrm{PSL}_2\mathbb{R}$ historically called *the geodesic flow* is defined by multiplying $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$ on the right, and its closed orbits are called *modular knots*. For each primitive hyperbolic element γ in $\mathrm{SL}_2\mathbb{Z}$, we may define the corresponding modular knot C_γ by $C_\gamma(t) = M_\gamma \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$ $(0 \leq 0 \leq \log \xi_\gamma)$, where $M_\gamma^{-1} \gamma M_\gamma = \left(\begin{smallmatrix} \xi_\gamma & 0 \\ 0 & \xi_\gamma^{-1} \end{smallmatrix}\right)$ with $\xi_\gamma > 1$. Every modular knot admits such a presentation. By virtue of Bonatti–Pinsky's nice compactification [BP20], we obtain the following:

Theorem 4 ([Uek21b, Theorem 4])**.** *Modular knots and the missing trefoil in S^3 obey the Chebotarev law, if ordered by length in a generic metric.*

As a corollary, we obtain a result on a function with arithmetic origin. The discriminant function $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ with $q = e^{2\pi\sqrt{-1}z}$, $z \in \mathbb{H}^2$ is a well-known modular function of weight 12. *The Dedekind symbol* Φ and *the Rademacher symbol* Ψ are the functions $\mathrm{SL}_2\mathbb{Z} \to \mathbb{Z}$ satisfying

$$\log\Delta(\gamma z) - \log\Delta(z) = \begin{cases} 6\log(-(cz+d)^2) + 2\pi i\Phi(\gamma) & \text{if } c \neq 0, \\ 2\pi i\Phi(\gamma) & \text{if } c = 0, \end{cases}$$

$$\Psi(\gamma) = \Phi(\gamma) - 3\mathrm{sgn}(c(a+d))$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2\mathbb{Z}$ acting on $z \in \mathbb{C}$ via the Möbius transformation $\gamma z = \frac{az+b}{cz+d}$. Here we take a branch of the logarithm so that $-\pi \leq \mathrm{Im}\log z < \pi$ holds. This Ψ factors through the conjugacy classes of $\mathrm{PSL}_2\mathbb{Z}$ and satisfies $\Psi(\gamma^{-1}) = -\Psi(\gamma)$ for any γ.

The Rademacher symbol Ψ is known to be a highly ubiquitous function. Indeed, Atiyah proved the equivalence of seven definitions rising from very distinct contexts [Ati87], whereas Ghys gave further characterizations ([BG92], [Ghy07, Sections 3.3–3.5], [DIT17, Appendix]), proving that *for each primitive hyperbolic* $\gamma \in \mathrm{SL}_2\mathbb{Z}$*, the linking number between the modular knot* C_γ *and the missing trefoil* K *coincides with the Rademacher symbol*, namely,

$$\mathrm{lk}(C_\gamma, K) = \Psi(\gamma)$$

holds. Theorem 4 for $\rho(\gamma) = \mathrm{lk}(C_\gamma, K) \bmod m$ together with some arguments yield the following.

Corollary 5 ([Uek21b, Corollary 9])**.** *Suppose that* γ *runs through primitive hyperbolic elements of* $\mathrm{SL}_2\mathbb{Z}$*. For any* $m \in \mathbb{Z}_{>0}$ *and* $k \in \mathbb{Z}/m\mathbb{Z}$*, we have*

$$\lim_{\nu\to\infty} \frac{\#\{\gamma \mid |\mathrm{tr}\gamma| < \nu,\ \Psi(\gamma) = k \text{ in } \mathbb{Z}/m\mathbb{Z}\}}{\#\{\gamma \mid |\mathrm{tr}\gamma| < \nu\}} = \frac{1}{m}.$$

The similar arguments may be applicable to other Fuchsian groups. Modular knots for triangle groups around any torus knot in S^3 will be finely studied in [MU21].

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