Analog of Menchov-Trokhimchuk theorem for monogenic functions in three-dimensional commutative algebra

Maxym Tkachuk
(Institute of Mathematics of NASU)
E-mail: maxim.v.tkachuk@gmail.com

Sergiy Plaksa
(Institute of Mathematics of NASU)
E-mail: plaksa62@gmail.com

Many scientists worked on finding new weaker conditions for holomorphicity of complex-valued functions: H. Bohr, H. Rademacher, D. Menchov [1], V. Fedorov, G. Tolstov, Y. Trokhimchuk [2, 3], G. Sindalovski, D.Teliakovski, E. Dolzhenko, M. Brodovich and their multidimencional generalizations: A. Bondar, V. Siryk, O. Gretskii.

Here is one of Menchov conditions: *function $F(\xi)$ satisfies K''' condition in point ξ_0 if exists limit*

$$\lim_{\xi\to\xi_0}\frac{F(\xi)-F(\xi_0)}{\xi-\xi_0}\,, \tag{1}$$

where ξ belongs to union of two noncollinear rays with common starting point ξ_0.

D. Menchov [1] has prowed that fulfillment of the condition K''' in any point of domain D (excluding not more than countable set) is sufficient for conformity of mapping F in case if $F : D \to \mathbb{C}$ is continuous univalent function. Y. Trokhimchuk [2] has removed univalency condition in following theorem.

Menchov-Trokhimchuk Theorem. *If function $F : D \to \mathbb{C}$ is continuous in domain D and in every its point, excluding not more than countable set, condition K''' is fulfilled, then function F is holomorphic in domain D.*

Analog of Menchov-Trokhimchuk Theorem for monogenic functions in space E_3.

Let $\mathbb{A}_3$ be 3-dimencional commutative associative algebra with unit 1 over the field $\mathbb{C}$ with basis $\{1,\rho,\rho^2\}$, such that $\rho^3=0$.

Let fix the real 3-dimencional subspace $E_3 := \{\zeta = xe_1 + ye_2 + ze_3 \;:\; x,y,z\in\mathbb{R}\}\subset\mathbb{A}_3$, where the vectors e_1,e_2,e_3 — are linearly independent over the real field $\mathbb{R}$, but, in general, not a basis of the algebra $\mathbb{A}_3$. Only one condition should be fulfilled: image of the E_3 under the mapping f is whole complex plane (see [5, 6]).

Function $\Phi'_G\colon\Omega\longrightarrow\mathbb{A}_3$ is called Gâteaux derivative of function $\Phi\colon\Omega\longrightarrow\mathbb{A}_3$, with domain $\Omega\subset E_3$, if in any point $\zeta\in\Omega$ exists element $\Phi'_G(\zeta)\in\mathbb{A}_3$ such that

$$\lim_{\delta\to 0+0}\left(\Phi(\zeta+\delta h)-\Phi(\zeta)\right)\delta^{-1} = h\Phi'_G(\zeta)\quad \forall h\in E_3\,. \tag{2}$$

Function $\Phi\colon\Omega\longrightarrow\mathbb{A}_3$ is called *monogenic* in domain $\Omega\subset E_3$, if Φ is continuous and has Gâteaux derivative in any point of Ω (see [8, 11]).

Intersection of radical of algebra $\mathbb{A}_3$ with linear space E_3 is the set of non-invertible elements which belongs to E_3. This set is the straight line $L := \{c\,l \,:\, c\in\mathbb{R}\}$, with direction vector $l\in E_3$. Preimage of any point $\xi\in\mathbb{C}$ in E_3 under the mapping f is the straight line $L^\zeta := \{\zeta + c\,l \,:\, c\in\mathbb{R}\}$, where ζ — element from E_3 such that $\xi = f(\zeta)$. Obviously, line L^ζ is parallel to line L.

Let domain $\Omega\subset E_3$ is convex in direction of straight line L (domain is called *convex in direction of straight line*, if it contains every segment joining two points of domain and parallel to this straight line).

Consider next hypercomplex analog of Menshov condition K''' in algebra $\mathbb{A}_3$ for functions $\Phi : \Omega \to \mathbb{A}_3$, defined in domain $\Omega \subset E_3$.

Definition 1. Let say, that function $\Phi : \Omega \to \mathbb{A}_3$ is fulfilled condition $K'''_{\mathbb{A}_3,E_3}$ at point $\zeta \in \Omega$, if exists element $\Phi_*(\zeta) \in \mathbb{A}_3$ such that equation

$$\lim_{\delta\to 0+0} \left(\Phi(\zeta+\delta h) - \Phi(\zeta)\right)\delta^{-1} = h\Phi_*(\zeta) \tag{3}$$

is fulfilled for three vectors h: h_1, h_2 and $h_3 = l$ or $h_3 = -l$, which are the basis of space E_3.

Theorem 2. *Let the domain $\Omega \subset E_3$ is convex in direction of straight line L, function $\Phi : \Omega \to \mathbb{A}_3$ is continuous in Ω and fulfill condition$K'''_{\mathbb{A}_3,E_3}$ in all points $\zeta \in \Omega$, except not more than countable set. Then:*

1) function Φ is monogenic in domain Ω;

2) function Φ extends to function monogenic in domain Π. This extension is unique and given by equality

$$\Phi(\zeta) = \frac{1}{2\pi i}\int\limits_{\gamma} \Big(F_0(\xi) + F_1(\xi)\rho + F_2(\xi)\,\rho^2\Big)(\xi-\zeta)^{-1}\,d\xi\,, \tag{4}$$

for all $\zeta \in \Pi$;

3) monogenic extension (4) of function Φ is differentiable in the sense of Lorch [10] *in Π.*

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