Maximal distance minimizers. Examples and properties

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I will talk about the sets which have the minimal length over the class of closed connected sets $\Sigma \subset \mathbb{R}^2$ satisfying the inequality $\max_{y\in M} \operatorname{dist}(y, \Sigma) \le r$ for a given compact set $M \subset \mathbb{R}^2$ and some given $r > 0$.

For a given compact set $M \subset \mathbb{R}^2$ consider the functional

$$F_M(\Sigma) := \sup_{y\in M} \operatorname{dist}(y, \Sigma),$$

where Σ is a closed subset of $\mathbb{R}^2$ and $\operatorname{dist}(y, \Sigma)$ stands for the Euclidean distance between y and Σ. Consider the class of closed connected sets $\Sigma \subset \mathbb{R}^2$ satisfying $F_M(\Sigma) \le r$ for some $r > 0$. We are interested in the sets of the minimal length (one-dimensional Hausdorff measure) $\mathcal{H}^1(\Sigma)$ over the mentioned class (*minimizers*).

It is known that for all $r > 0$ the set of minimizers is nonempty. It is proven also that for each minimizer of positive length the equality $F_M(\Sigma) = r$ holds. Furthermore the set of minimizers coincides with the set of solutions of the dual problem: minimize F_M over all closed connected sets $\Sigma \subset \mathbb{R}^2$ with prescribed bound on the total length $\mathcal{H}^1(\Sigma) \le l$.

In [1] (for the plane) and in [2] (for the general case) some properties of minimizers have been proven:

(a) A minimizer cannot contain loops (homeomorphic images of circles).
(b) For every point $x \in \Sigma$ one of two statements is true:
  i there exists a point $y \in M$ (may be not unique) such that $\operatorname{dist}(x, y) = r$ and $B_r(y)\cap\Sigma = \emptyset$;
  ii there exists an $\varepsilon > 0$ such that $S_\Sigma \cap B_\varepsilon(x)$ is either a segment or a regular tripod, i.e. the union of three segments with an endpoint in x and relative angles of $2\pi/3$.

The minimizers for some sets M are known (see pictures) although usually this is not an easy task. Recently (see [3]) at the plane the regularity of minimizers was proved.

Theorem 1. *Let Σ be a maximal distance minimizer for a compact set $M \subset \mathbb{R}^2$. Then*

(i) *Σ is a union of a finite number of arcs (injective images of the segment $[0; 1]$).*
(ii) *The angle between each pair of tangent rays at every point of Σ is greater or equal to $2\pi/3$. The number of tangent rays at every point of Σ is not greater than 3. If it is equal to 3, then there exists such a neighbourhood of x that the arcs in it coincide with line segments.*

References

[1] Miranda, Jr., M. and Paolini, E. and Stepanov, E., *On one-dimensional continua uniformly approximating planar sets.* Volume 27 of *Calculus of Variations and Partial Differential Equations*, 2006.

[2] Paolini, E. and Stepanov, E., *Qualitative properties of maximum distance minimizers and average distance minimizers in $\mathbb{R}^n$*, volume 122 of *Journal of Mathematical Sciences.* New York, 2004.

[3] Yana Teplitskaya. *On regularity of maximal distance minimizers*, arXiv preprint arXiv:1910.07630, 2019.

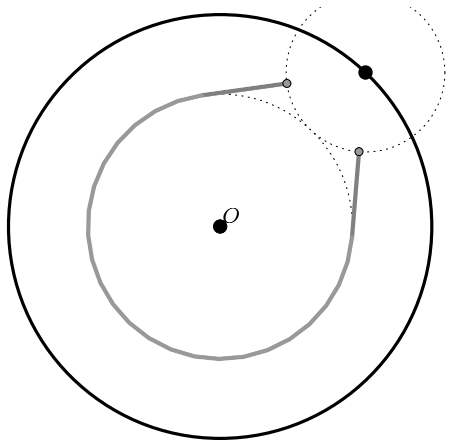

FIGURE 1. An example where $M = \partial B_R(O)$, where $R > 4.98r$.

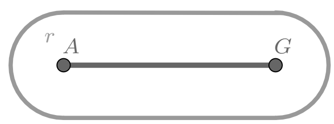

FIGURE 2. An example where $M := \partial B_r([AG])$, $\Sigma = [AG]$.

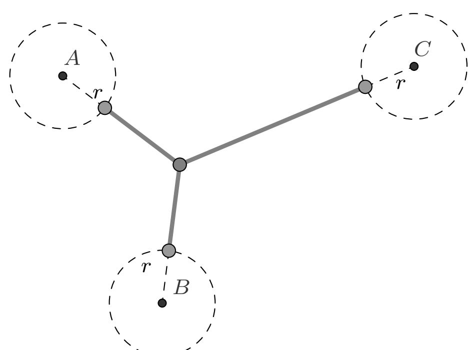

FIGURE 3. An example where $M := \{A, B, C\}$, Σ is a tripod.