Projective invariants of linear planar 3-webs

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In this talk we present projective differential invariants of linear planar 3-webs. Linear 3-web on the plane $\mathbb{R}^2(x, y)$ is an unordered set of 3 linear foliations with the condition that leaves of any pair of foliations are transversal to each other. Any such web is defined by the set of 3 solutions $w = w(x, y)$ to the Euler equation (see [1])

$$w_y = ww_x.$$

We will consider actions of the group of projective transformations $SL_3(\mathbb{R})$ of the plane. This actions carries over to the space of solutions of the Euler equation. Representations of the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ by vector fields are

$$X_A = (a_{13} + (a_{11} - a_{33})x + a_{12}y - a_{32}xy - a_{33}x^2)\partial_x + \\ + (a_{23} + a_{21}x + (a_{22} - a_{33})y - a_{31}xy - a_{32}y^2)\partial_y,$$

where the matrix $A = \|a_{ij}\|_{i,j=1,2,3} \in \mathfrak{sl}_3(\mathbb{R})$.

Proposition 1. *The vector fields*

$$\overline{X}_A = X_A + \lambda_A(w)\partial_w,$$

where

$$\lambda_A(w) = (a_{21} - a_{31}y)w^2 + (a_{11} - a_{22} - a_{31}x + a_{32}y)w + a_{32}x - a_{12},$$

define representations of the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ on the total space of the bundle

$$\pi' : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2, \quad \pi'(x, y, w) \mapsto (x, y).$$

Moreover, the vector fields $\overline{X}_A$ are symmetries of the Euler equation.

Linear planar 3-webs are defined by a set of solutions w^1, w^2, w^3 to the Euler equation.

Proposition 2. *The vector fields*

$$\overline{X}_A = X_A + \lambda_A(w^1)\partial_{w^1} + \lambda_A(w^2)\partial_{w^2} + \lambda_A(w^3)\partial_{w^3}$$

define a representation of Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ on the total space of the bundle

$$\pi : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2, \quad \pi(x, y, w^1, w^2, w^3) \mapsto (x, y).$$

Moreover, the vector fields $\overline{X}_A$ are symmetries of the system of Euler equations

$$w^1_y = ww^1_x, \quad w^2_y = w^2w^2_x, \quad w^3_y = w^3w^3_x. \tag{1}$$

System of equations (1) defines the submanifold

$$E_1 \subset J^1(\pi), \quad E_1 = \{w^1w^1_x - w^1_y = 0, w^2w^2_x - w^2_y = 0, w^3w^3_x - w^3_y = 0\},$$

where $J^1(\pi)$ is the bundle of 1-jets of sections of this bundle. Let $E_k \subset J^k(\pi)$ be a kth prolongation of this manifold.

A rational function I on the manifold E_k is called *a projective differential invariant* of linear 3-webs of order $\leq k$, if $\overline{X}^{(k)}_A(I) = 0$ on E_k for all $A \in \mathfrak{sl}_3(\mathbb{R})$. Here $\overline{X}^{(k)}_A$ is the kth prolongation of the vector field $\overline{X}_A$.

Solving the system of equations $\overline{X}^{(2)}_A(I) = 0$, we get the following result.

Theorem 3. *The field of rational projective differential invariants of order ≤ 2 of linear 3-webs is generated by invariants of order 2*

$$I_{21} = \frac{w^2_{xx}}{w^3_{xx}}, \quad I_{22} = -\frac{(-w^2+w^3)w^1_x + (w^1-w^3)w^2_x - w^3_x(w^1-w^2)}{\sqrt{w^3_{xx}(w^2-w^3)(w^1-w^3)(w^1-w^2)}}, \quad I_{23} = \frac{w^1_{xx}}{w^3_{xx}}.$$

This field separates regular $SL_3(\mathbb{R})$-orbits in E_2.

To describe the field of all projective differential invariants of linear 3-webs, we use the Lie-Tresse theorem (see [2]).

Theorem 4. *The field of rational projective differential invariants of linear 3-webs is generated by the basis invariants I_{21}, I_{22}, I_{23} and the invariant derivations*

$$\nabla_1 = -\frac{(-w^2+w^3)w^1}{w^1w^2_x - w^1w^3_x - w^1_xw^2 + w^1_xw^3 + w^2w^3_x - w^2_xw^3}\frac{d}{dx}$$
$$+\frac{-w^2+w^3}{w^1w^2_x - w^1w^3_x - w^1_xw^2 + w^1_xw^3 + w^2w^3_x - w^2_xw^3}\frac{d}{dy},$$
$$\nabla_2 = -\frac{(w^3-w^1)w^2}{w^1w^2_x - w^1w^3_x - w^1_xw^2 + w^1_xw^3 + w^2w^3_x - w^2_xw^3}\frac{d}{dx}$$
$$+\frac{-w^2+w^3}{w^1w^2_x - w^1w^3_x - w^1_xw^2 + w^1_xw^3 + w^2w^3_x - w^2_xw^3}\frac{d}{dy}.$$

This field separates regular orbits.

References

[1] V.V. Goldberg, V.V. Lychagin. Geodesic webs on a two-dimensional manifold and Euler equations. *Acta Applicandae Mathematicae*, 04/2012; 109(1):5–17. DOI 10.1007/s10440-009-9437-1.

[2] B.S Kruglikov, V.V. Lychagin. Global Lie-Tresse theorem. *Selecta Mathematica*, 02/2016. DOI 10.1007/s00029-015-0220-z.