Asymptotic estimates for the widths of classes of periodic functions of high smothness

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Let L_p, $1 \le p \le \infty$, and C be the spaces of 2π–periodic functions with standart norms $\|\cdot\|_p$ and $\|\cdot\|_C$, respectively.

Denote by $C^{\psi}_{\bar{\beta},p}$, $1 \le p \le \infty$, the set of all 2π-periodic functions f, representable as convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \varphi(x-t)\Psi_{\bar{\beta}}(t)dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B_p^0 = \{g \in L_p : \ \|g\|_p \le 1, \ g \perp 1\}, \qquad (1)$$

with a fixed generated kernel $\Psi_{\bar{\beta}} \in L_{p'}$, $1/p + 1/p' = 1$, the Fourier series of which has the form

$$S[\Psi_{\bar{\beta}}](t) = \sum_{k=1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta_k \pi}{2}\right), \quad \beta_k \in \mathbb{R}, \quad \psi(k) \ge 0. \qquad (2)$$

A function f in the representation (1) is called $(\psi, \bar{\beta})$-integral of the function φ and is denoted by $\mathcal{J}^{\psi}_{\bar{\beta}}\varphi$ ($f = \mathcal{J}^{\psi}_{\bar{\beta}}\varphi$). If $\psi(k) \ne 0$, $k \in \mathbb{N}$, then the function φ in the representation (1) is called $(\psi, \bar{\beta})$-derivative of the function f and is denoted by $f^{\psi}_{\bar{\beta}}$ ($\varphi = f^{\psi}_{\bar{\beta}}$). The concepts of $(\psi, \bar{\beta})$-integral and $(\psi, \bar{\beta})$-derivative was introduced by Stepanets (see [4]). Since $\varphi \in L_p$ and $\Psi_{\bar{\beta}} \in L_{p'}$, then the function f of the form (1) is a continuous function, i.e. $C^{\psi}_{\bar{\beta},p} \subset C$ (see [4, Proposition 3.9.2.]).

In the case $\beta_k \equiv \beta$, $\beta \in \mathbb{R}$, the classes $C^{\psi}_{\bar{\beta},p}$ are denoted by $C^{\psi}_{\beta,p}$. For $\psi(k) = k^{-r}, r > 0$, the classes $C^{\psi}_{\bar{\beta},p}$ and $C^{\psi}_{\beta,p}$ are denoted by $W^r_{\bar{\beta},p}$ and $W^r_{\beta,p}$, respectively. The classes $W^r_{\beta,p}$ are the well-known Weyl-Nagy classes. For $\psi(k) = e^{-\alpha k^r}$, $\alpha > 0$, $r > 0$, the classes $C^{\psi}_{\bar{\beta},p}$ and $C^{\psi}_{\beta,p}$ are denoted by $C^{\alpha,r}_{\bar{\beta},p}$ and $C^{\alpha,r}_{\beta,p}$, respectively. The sets $C^{\alpha,r}_{\beta,p}$ are well-known classes of the generalized Poisson integrals.

Let $\mathfrak{N}$ be some functional class from the space C ($\mathfrak{N} \subset C$). The quantity

$$E_n(\mathfrak{N})_C = \sup_{f \in \mathfrak{N}} E_n(f)_C = \sup_{f \in \mathfrak{N}} \inf_{T_{n-1} \in \mathcal{T}_{2n-1}} \|f - T_{n-1}\|_C \qquad (3)$$

is called the best uniform approximation of the class $\mathfrak{N}$ by elements of the subspace $\mathcal{T}_{2n-1}$ of trigonometric polynomials T_{n-1} of the order $n-1$.

The order estimates for the best approximations $E_n(K)_C$ of classes $K = C^{\psi}_{\bar{\beta},p}$, $1 \le p \le \infty$, (and, hence, classes $W^r_{\beta,p}$, $C^{\alpha,r}_{\beta,p}$ and $C^{\psi}_{\beta,p}$) depending on rate of decreasing to zero of sequences $\psi(k)$ were obtained, in particular, in the works of Temlyakov (1993), Hrabova and Serdyuk (2013), Serdyuk and Stepanyuk (2014) etc.

If the sequences $\psi(k)$ decrease to zero faster than any geometric progression, then asymptotic equations of the best uniform approximations are even known (see [3] and the bibliography available there).

In [3] it was shown that for such classes $C^{\psi}_{\bar\beta,p}$ the following asymptotic equations take places

$$E_n(C^{\psi}_{\bar\beta,p})_C \sim \mathcal{E}_n(C^{\psi}_{\bar\beta,p})_C \sim \frac{\|\cos t\|_{p'}}{\pi}\psi(n), \quad 1\le p\le\infty, \quad \frac{1}{p}+\frac{1}{p'}=1, \tag{4}$$

where $\mathcal{E}_n(C^{\psi}_{\bar\beta,p}) = \sup\limits_{f\in C^{\psi}_{\bar\beta,p}} \|f-S_{n-1}(f)\|_C$, $S_{n-1}(f)$ is the partial Fourier sum of order $n-1$ of the function f, and $A(n)\sim B(n)$ as $n\to\infty$ means that $\lim\limits_{n\to\infty} A(n)/B(n)=1$.

For $p=\infty$ in the case of $K=W^r_{\bar\beta,\infty}, r>0$, and in the cases of $K=C^{\alpha,r}_{\bar\beta,\infty}, r\ge 1$, and $K=C^{\psi}_{\bar\beta,\infty}$ ($K=C^{\psi}_{\beta,\infty}$) for certain restrictions on sequences ψ and $\bar\beta$ the exact values of the best uniform approximations are known thanks to the works of Favard (1936, 1937), Akhiezer and Krein (1937), Krein (1938), Nagy (1938), Stechkin (1956), Dziadyk (1959, 1974), Sun (1961), Bushanskij (1978), Pinkus (1985), Serdyuk (1995, 1999, 2002) etc.

For $p=2$ and for arbitrary $\bar\beta=\beta_k\in\mathbb{R}$, $\sum\limits_{k=1}^{\infty}\psi^2(k)<\infty$ the exact values for the quantity $\mathcal{E}_n(C^{\psi}_{\bar\beta,2})_C$ are also known (see [2]).

Let K be a convex centrally symmetric subset of C and let $b_N(K,C), d_N(K,C), \lambda_N(K,C)$, and $\pi_N(K,C)$ be Bernstein, Kolmogorov, linear, and projection N-widths of the set K in the space C [1].

The results containing order estimates of the widths b_N, d_N, λ_N or π_N in the case of $K=C^{\psi}_{\bar\beta,p}$ (and, in particular, $W^r_{\beta,p}$ and $C^{\psi}_{\beta,p}$) can be found, for example, in the works of Tikhomirov (1976), Pinkus (1985), Kornejchuk (1987), Kashin (1977), Kushpel' (1989), Temlyakov (1990, 1993) etc.

Theorem 1. *Let $\{\beta_k\}_{k=1}^{\infty}$, $\beta_k\in\mathbb{R}$, and $\psi(k)>0$ satisfies the condition $\sum\limits_{k=1}^{\infty}\psi^2(k)<\infty$. Then for all $n\in\mathbb{N}$ the following inequalities hold*

$$\frac{1}{\sqrt{\pi}}\left(\frac{1}{\psi^2(n)}+2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{-\frac12} \le P_{2n}(C^{\psi}_{\bar\beta,2},C)\le P_{2n-1}(C^{\psi}_{\bar\beta,2},C)\le \frac{1}{\sqrt{\pi}}\left(\sum_{k=n}^{\infty}\psi^2(k)\right)^{\frac12}, \tag{5}$$

where P_N is any of the widths b_N, d_N, λ_N or π_N.

If, in adition, $\psi(k)$ satisfies the condition $\lim\limits_{n\to\infty}\max\left\{\psi(n)\left(\sum\limits_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac12}, \frac{1}{\psi(n)}\left(\sum\limits_{k=n+1}^{\infty}\psi^2(k)\right)^{\frac12}\right\}=0,$

then the following asymptotic equalities hold

$$\left.\begin{array}{l} P_{2n}(C^{\psi}_{\bar\beta,2},C) \\ P_{2n-1}(C^{\psi}_{\bar\beta,2},C)\end{array}\right\} = \psi(n)\left(\frac{1}{\sqrt{\pi}}+\mathcal{O}(1)\max\left\{\psi(n)\left(\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac12}, \frac{1}{\psi(n)}\left(\sum_{k=n+1}^{\infty}\psi^2(k)\right)^{\frac12}\right\}\right), \tag{6}$$

where $\mathcal{O}(1)$ are the quantities uniformly bounded in all parameters.

The equalities (6) are realized by trigonometric Fourier sums $S_{n-1}(f)$.

References

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