Painlevé VI Solutions From Equivariant ADHM Instanton Bundles

Jan Segert
(Mathematics Department, University of Missouri, Columbia MO, USA)
E-mail: `segertj@missouri.edu`

We report on the paper [4] . Hitchin [3] had produced a pair of solutions $\lambda_0^{\pm}$ for the Painlevé VI differential equation from an $SL_2(\mathbb{C})$ action on the trivial bundle $E_0 \to P^3$ over complex projective space. We generalize to produce PVI solutions $\lambda_m^{\pm}$ for each nonnegative integer m from $SL_2(\mathbb{C})$ actions on the equivariant instanton bundles $E_m \to P^3$ constructed in [2] via an equivariant version of the Atiyah-Drinfeld-Hitchin-Manin construction [1].

Theorem 1. *For each nonnegative integer m, the equivariant instanton bundle E_m yields a pair of explicitly computable algebraic Painlevé VI solutions $\lambda_m^{\pm}(t)$, expressed implicitly in terms of the rational function*

$$t(w) = \frac{(1+w)\ (-3+w)^3}{(-1+w)\ (3+w)^3}$$

and a rational function of the form

$$\lambda_m^{\pm}(w) = \left(\frac{(-3+w)^2}{(-1+w)\ (3+w)} \right) \frac{(-1+w^2)\, f_m^{\pm}(w) + 8\, g_m^{\pm}(w)}{(3+w^2)\, f_m^{\pm}(w) - 24\, g_m^{\pm}(w)},$$

where $f_m^{\pm}$ and $g_m^{\pm}$ are even polynomials of degree at most $2m(m+1)$.

We have found explicit Okamoto transfromations $Q^{\pm 1}$ relating the two hierarchies of solutions $\lambda_m^{\pm}$ in a manner reminiscent of the familiar *creation operators* for eigenstates of the quantum harmonic oscillator. The following was proved case-by-case for a finite number of nonnegative integers m, and conjectured to hold for all nonnegative integers m:

Theorem 2. *For each nonnegative integer $m \leq 4$,*

$$\lambda_m^{+} = Q^m \lambda_0^{+}, \qquad \lambda_m^{-} = Q^{-m} \lambda_0^{-}.$$

We interpret each "creation operator" $Q^{\pm 1}$ as a "shadow" of a putative creation operator for equivariant instanton bundles E_m, which is indicated by the dashed arrows in the summary diagram:

$$\begin{tikzcd}
\lambda_0^{+} \arrow[r, "Q"] & \lambda_1^{+} \arrow[r, "Q"] & \lambda_2^{+} \arrow[r, "Q"] & \lambda_3^{+} \arrow[r, "Q"] & \lambda_4^{+} \\
E_0 \arrow[u] \arrow[d] \arrow[r, dashed] & E_1 \arrow[u] \arrow[d] \arrow[r, dashed] & E_2 \arrow[u] \arrow[d] \arrow[r, dashed] & E_3 \arrow[u] \arrow[d] \arrow[r, dashed] & E_4 \arrow[u] \arrow[d] \\
\lambda_0^{-} \arrow[r, "Q^{-1}"] & \lambda_1^{-} \arrow[r, "Q^{-1}"] & \lambda_2^{-} \arrow[r, "Q^{-1}"] & \lambda_3^{-} \arrow[r, "Q^{-1}"] & \lambda_4^{-}
\end{tikzcd} .$$

References

[1] M.F. Atiyah, V.G. Drinfeld, N.J. Hitchin, and Yu.I. Manin. Construction of Instantons. Phys. Lett. **65A** 185 (1978)
[2] Gil Bor and Jan Segert. Symmetric instantons and the ADHM construction. Commun. Math. Phys **183** 183–203 (1997)
[3] Nigel J. Hitchin. Twistor spaces, Einsten metrics and isomonodromic deformations. J. Diff. Geom. **42** 30–112 (1995)
[4] Jan Segert. Okamoto Transformations relating Equivariant Instanton Bundles via Painlevé VI Preprint (2021), `arXiv:2103.06321 [math.DG]`