On the metric equations generated by symplectic deformations on $P_2(C)$

Anatolij Prykarpatsky
(Drohobych Pedagogical University, Lviv region, Ukraine, and the Department of Computer Science at the Cracov University of Technology, Kraków 31-155, Poland)
E-mail: `pryk.anat@cybergal.com`

Ihor Mykytyuk
(Pidstryhach Institute for Applied Problems of Mechanics and Mathematics at the NAS, Lviv, 79060, Ukraine, and the Department of Computer Science at the Cracov University of Technology, Kraków, 31-155, Poland)
E-mail: `mykytyuk.i@gmail.com`

We analyzed in detail the cohomology structure of the symplectic form deformation and applied recently developed generalized transformations which were suggested in the classical works by Enneper and Weierstrass about one and half century ago and succeeded in reformulating the "symplectic" modification of the Monge-Ampere equation by means of specially constructed coordinates, related with the natural projector expansion on $T(P_2(\mathbb{C}))$ and found its special solutions. Let us consider a compact complex n-dimensional manifold M^n, endowed with the Kähler symplectic form $\omega \in \Lambda^2(M^n)$ and define the related Monge-Ampere equation, describing a deformation of this symplectic structure:

$$(\omega + i\partial\bar{\partial}\varphi)^n = (\exp f)\,\omega^n \tag{1}$$

under the normalizing conditions

$$\int_{M^n} (\exp f)\,\omega^n = \int_{M^n} \omega^n, \qquad \int_{M^n} \varphi\omega^n = 0, \tag{2}$$

where $\varphi \in C^\infty(M^n;\mathbb{R})$ is a real valued function on M^n and $\bar{\partial}$ is the complex ∂-bar differential, corresponding to the standard differential splitting $d = \partial \oplus \bar{\partial} : \Lambda(M^n) \to \Lambda(M^n)$ on the complex manifold M^n. In a general case it was supposed [16] that if the two-form $\left(\omega + i\partial\bar{\partial}\varphi\right) \in \Lambda^2(M^n)$ is real valued and the first Chern class $c_1(M^n) = 0$ of a Kähler manifold M^n, then there exists a Riemannian metric $g : T(M^n) \times T(M^n) \to \mathbb{R}$ of the Calabi-Yau type, whose holonomy group [5, 8] coincides with a subgroup of the Lie group SU(2), generating, in particular, a so called Einsteinian metric. The equation (1) is always [16] solvable, yet its holonomy groups, in general, not classified and its unitarity remains to be open.

We here also remark that there exists a slightly different modified Monge-Ampere type deformation equation

$$(\omega + dJ^*d\varphi)^n = (\exp f)\,\omega^n, \tag{3}$$

on a real symplectic manifold $\bar{M}^{2n} \simeq M^n$, where $f \in C^\infty(\bar{M}^{2n};\mathbb{R})$ and $J : T(\bar{M}^{2n}) \to T(\bar{M}^{2n}), J^2 = -I$, is a suitably chosen nonintegrable quasi-complex structure on the manifold $\bar{M}^{2n}$ and $J^* : T^*(\bar{M}^{2n}) \to T^*(\bar{M}^{2n})$ denotes its conjugate. It was proved [2] that if the structure $J : T(\bar{M}^{2n}) \to T(\bar{M}^{2n})$ is integrable, then the equation (3) reduces to the Monge-Ampere equation (1) on the related complex manifold $M^n \simeq \bar{M}^{2n}$ owing to the classical Newalander-Nirenberg [10] criterion. Otherwise, if the equation (3) is solvable for its arbitrarily chosen right hand side, then the quasi-complex structure $J : T(\bar{M}^{2n}) \to T(\bar{M}^{2n})$ proves to be necessary [2, 9, 11] a complex one, once more reducing the equation (3) to the Monge-Ampere equation (1).

In our note we are interested in the following "symplectic" modification

$$(\omega + dd^s\varphi)^2 = (\exp f)\,\omega^2 \tag{4}$$

of the Monge-Ampere (1) on the complex Kähler manifold $M^2 = P_2(\mathbb{C})$, where $\varphi \in \Lambda^2(M^2)$ is a searched for two-form and $d^s := (-1)^{k+1} \star_s d\star_s, dd^s = -d^s d,$ denotes the symplectic Hodge type differentiation. It is well known that any compact two-dimensional Kähler manifold M^2 with the Chern class $c_1(M^2) = 0$ is hiper-Kähler, possessing exactly three Kähler fundamental forms ω_I, ω_J and $\omega_K \in \Lambda^2(\bar{M}^4)$, corresponding to three complex structures I, J and $K : T(\bar{M}^4) \to T(\bar{M}^4)$. As for the compact projective two-dimensional Kähler manifold $M^2 = P_2(\mathbb{C})$ the Chern class $c_1(M^2) \neq 0$, it is not hiper-Kähler, its holomorphic volume two- form is not composed of the symplectic forms ω_J and $\omega_K \in \Lambda^2(\bar{M}^4)$. Notwithstanding this fact, based on the equalities (**??**) and the well known [1, 14, 15] relationship

$$\star_s \eta = -\eta \tag{5}$$

for an arbitrary "primitive" holomorphic volume two-form $\eta \in \Lambda^2_{hol}(M^2)$, satisfying the additional condition $\eta \wedge \omega = 0$, one easily derives that for any two cohomological "primitive" holomorphic volume two-forms Ω_1 and $\Omega_2 \in \Lambda^2_{hol}(M^2)$ there holds the following interesting relationship:

$$\Omega_1 - \Omega_2 = dd^s \psi \tag{6}$$

for some smooth two-from $\psi \in \Lambda^2(M^2)$, solving the problem (4) for the case when the symplectic structure $\omega \in \Lambda^2(M^2)$ is replaced by a holomorphic volume form $\Omega \in \Lambda^2_{hol}(M^2)$. Having analyzed in detail the cohomology structure of the two-form expression $(\omega + dd^s\varphi) \in \Lambda^2(M^2)$ and applied generalized transformations which were suggested in the classical works by Enneper [4] and Weierstrass [13] about one and half century ago and recently developed in [6], we succeeded in reformulating the "symplectic" modification of the Monge-Ampere (4) by means of specially constructed coordinates, related with the natural projector expansion from in $P_2(\mathbb{C})$ and find its special solutions.

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