On the matrix equation $AX - YB = C$ over Bezout domains

Volodymyr Prokip
(IAPMM NAS of Ukraine, Str. Naukova 3b, L'viv, Ukraine, 79060)
E-mail: v.prokip@gmail.com

Let $\mathrm{R}_{m,n}$ be the set of $m \times n$ matrices over a commutative ring R with identity $e \neq 0$. Denote by I_n the identity $n \times n$ matrix and by $0_{m,n}$ the zero $m \times n$ matrix. For any matrix $A \in \mathrm{R}_{m,n}$ A^t denotes the transpose of A. We will denote by $GL(m, \mathrm{R})$ the set of invertible matrices in $\mathrm{R}_{m,m}$. We will write $C_{\downarrow i}$ for the the i-th column of the matrix $C \in \mathrm{R}_{m,n}$ and $\mathrm{vec}\,(C)$ will denote an ordered stock of columns of C, i.e.,

$$\mathrm{vec}\,(C) = \begin{bmatrix} C_{\downarrow 1} \\ C_{\downarrow 2} \\ \vdots \\ C_{\downarrow n} \end{bmatrix}.$$

In this note we present alternative methods for finding solutions of the Sylvester matrix equation

$$AX - YB = C, \tag{1}$$

where A, B and C are given matrices of suitable sizes over a commutative domain.

This equation has been considered by several authors including Roth [12] over a field, Hartwig [6] over a regular ring, Gustafson [5] over a commutative ring with identity, Emre and Silverman [3] over a polynomial ring, Özgüler [7] over a principal ideal domain, Dajić [2] over an associative ring with unit. In general, Gustafson [5] has proved that equation (1) over a commutative ring R with identity has a solution (X, Y) over R if and only if the matrices $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are equivalent. This is a generalization of Roth's result [12], which gives the same criterion for the case, where R is a field. Similar considerations on solvability of equation (1) can be found in original paper [1].

1. Let R be a Bezout domain. Without reducing the generality we will assume that $A \in \mathrm{R}_{m,m}$, $B \in \mathrm{R}_{n,n}$ and $C \in \mathrm{R}_{m,n}$, and X, Y are unknown $m \times n$ matrices over R. Using the Kronecker product matrix equation (1) may be considered in the form of equivalent linear system (see [4])

$$(I_n \otimes A)\,\mathrm{vec}\,(X) - \left(B^t \otimes I_m\right)\mathrm{vec}\,(Y) = \mathrm{vec}\,(C).$$

Theorem 1. *Matrix equation (1) over Bezout domain* R *is solvable if and only if matrices*

$$\begin{bmatrix}(I_n \otimes A) & \left(B^t \otimes I_m\right) & 0_{mn,1}\end{bmatrix} \quad \textit{and} \quad \begin{bmatrix}(I_n \otimes A) & \left(B^t \otimes I_m\right) & \mathrm{vec}\,(C)\end{bmatrix}$$

are column equivalent, i.e., the right Hermite normal forms of these matrices are the same.

Corollary 2. *Let $A_i \in \mathrm{R}_{m,m}$, $B_i \in \mathrm{R}_{n,n}$ and $C_i \in \mathrm{R}_{m,n}$, $i = 1, 2$. Matrix equations $A_1X - YB_1 = C_1$ and $A_2X - YB_2 = C_2$ have a common solution over Bezout domain* R *if and only if matrices*

$$\begin{bmatrix}(I_n \otimes A_1) & \left(B_1^t \otimes I_m\right) & 0_{mn,1} \\ (I_n \otimes A_2) & \left(B_2^t \otimes I_m\right) & 0_{mn,1}\end{bmatrix} \quad \textit{and} \quad \begin{bmatrix}(I_n \otimes A_1) & \left(B_2^t \otimes I_m\right) & \mathrm{vec}\,(C_1) \\ (I_n \otimes A_2) & \left(B_2^t \otimes I_m\right) & \mathrm{vec}\,(C_2)\end{bmatrix}$$

are column equivalent, i.e., the right Hermite normal forms of these matrices are the same.

We were using results of papers [9] and [10] for proving Theorem 1.

2. In this parch R is a principal ideal domain. We denote by (a, b) the greatest common divisor of nonzero elements $a, b \in \mathrm{R}$. Let $A \in \mathrm{R}_{m,m}$ and $\mathrm{rank}\, A = r$. For the matrix A there exist matrices $U, V \in GL(m, \mathrm{R})$ such that $UAV = S_A = \mathrm{diag}(a_1, a_2, \ldots, a_r, 0, \ldots, 0)$ is the Smith normal form of A.

Theorem 3. *Let* $A \in \mathrm{R}_{m,m}$*,* $B \in \mathrm{R}_{n,n}$*,* $C \in \mathrm{R}_{m,n}$ *and* $\mathrm{rank}A = p$*,* $\mathrm{rank}B = q$*. Further, let* $U_A, V_A \in GL(m, \mathrm{R})$ *and* $U_B, V_B \in GL(m, \mathrm{R})$ *such that*

$$U_A A V_A = S_A = \mathrm{diag}(a_1, a_2, \dots, a_p, 0, \dots, 0), \quad U_B B V_B = S_B = \mathrm{diag}(b_1, b_2, \dots, b_q, 0, \dots, 0).$$

Matrix equation (1) is solvable over R *if and only if*

$$U_A C V_B = \left[\begin{array}{ccc|c} f_{11} & \dots & f_{1q} & \\ \vdots & \ddots & \vdots & 0_{p,n-q} \\ f_{p1} & \dots & f_{pq} & \\ \hline & 0_{m-p,q} & & 0_{m-p,n-q} \end{array}\right] \quad \textit{and} \quad (a_i, b_j) | f_{ij} \; (\textit{divides})$$

for all $i = 1, 2, \dots, p$ *i* $j = 1, 2, \dots, q$*.*

It is clear that if matrices $A \in \mathrm{R}_{m,m}$ and $B \in \mathrm{R}_{n,n}$ are nonsingular and $(\det A, \det B) = e$, then matrix equation (1) is solvable for an arbitrary matrix $C \in \mathrm{R}_{m,n}$.

Suppose that matrix equation (1) is solvable under the conditions of Theorem 3. Then for invariant factors a_i and b_j of matrices A and B respectively there exist $\alpha_{ij}, \beta_{ij} \in \mathrm{R}$ such that $a_i \alpha_{ij} - \beta_{ij} b_j = f_{ij}$ for all $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$. Put

$$X_\alpha = \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1q} \\ \vdots & \ddots & \vdots \\ \alpha_{p1} & \dots & \alpha_{pq} \end{bmatrix} \quad \text{and} \quad Y_\beta = \begin{bmatrix} \beta_{11} & \dots & \beta_{1q} \\ \vdots & \ddots & \vdots \\ \beta_{p1} & \dots & \beta_{pq} \end{bmatrix}.$$

Then for arbitrary matrices $P_{12}, Q_{12} \in \mathrm{R}_{p,n-q}$, $P_{21}, Q_{21} \in \mathrm{R}_{m-p,q}$ and $P_{22}, Q_{22} \in \mathrm{R}_{m-p,n-q}$ the pair of matrices

$$X_P = V_A^{-1} \begin{bmatrix} X_\alpha & P_{12} \\ P_{21} & P_{22} \end{bmatrix} V_B^{-1} \quad \text{and} \quad Y_Q = U_A^{-1} \begin{bmatrix} Y_\beta & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} U_B^{-1}$$

is the general solution of matrix equation (1). We note that Theorem 3 can be used for finding solutions with some properties of equation (1) (see [8] and [11]).

References

[1] D.S. Cvetković-Ilić, J.N. Radenković, Qing-Wen Wang. Algebraic conditions for the solvability to some systems of matrix equations. *Linear Multilinear Algebra*, 1–31, 2019. doi.org/10.1080/03081087.2019.1633993

[2] A. Dajić. Common solutions of linear equations in a ring with applications. *Electron. J. Linear Algebra*, 30 : 66–79, 2015. doi.org/10.13001/1081-3810.2910

[3] E. Emre, L.M. Silverman. The equation $XR + QY = \Phi$: a characterization of solutions. *SIAM J. Control Optim.*, 19(1) : 33–38, 1981. doi.org/10.1137/0319003

[4] A. Graham. *Kronecker products and matrix calculus with applications.* Courier Dover Publications. New York, 2018.

[5] W.H. Gustafson. Roth's theorems over commutative rings. *Linear Algebra and Appl.*, 23 : 245–251, 1979.

[6] R.E. Hartwig. Roth's equivalence problem in unit regular rings. *Proc. Amer. Math. Soc.*, 59(1) : 39–44, 1976.

[7] A.B. Özgüler. The equation $AXB + CYD = E$ over a principal ideal domain. *SIAM J. Matrix Analysis Appl.*, 12(3) : 581–591, 1991. doi.org/10.1137/0612044

[8] V.M. Prokip. About the uniqueness solution of the matrix polynomial equation $A(\lambda)X(\lambda) - Y(\lambda)B(\lambda) = C(\lambda)$. *Lobachevskii J. Math.*, 29 : 186–190, 2008. doi.org/10.1134/S1995080208030098

[9] V.M. Prokip. On the solvability of a system of linear equations over the domain of principal ideals. *Ukrainian Math. J.*, 66 : 633–637, 2014. doi.org/10.1007/s11253-014-0960-5

[10] V.M. Prokip. On the solvability of a system of matrix equations $AX = B$ and $BY = A$ over associative rings. *J. Math. Sci.*, 238 : 22–31, 2019. doi.org/10.1007/s10958-019-04215-y

[11] V.M. Prokip. On the divisibility of matrices with remainder over the domain of principal ideals. *J. Math. Sci.*, 243 : 45–55, 2019. doi.org/10.1007/s10958-019-04524-2

[12] W.E. Roth. The equations $AX - XB = C$ and $AX - YB = C$ in matrices. *Proc.Amer. Math. Soc.*, 3(3) : 392–396, 1952. doi.org/10.2307/2031890