Morse flows with singularities on boundary of 3-manifolds

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A flow X on a manifold with boundary ∂M is called *a Morse flow* if it satisfies the following conditions:

(1) the set of non-wandering points $\Omega(X)$ has a finite number of points that all are hyperbolic;
(2) if $u, v \in \Omega(X)$, then the unstable manifold $W^u(u)$ is transverse to the stable manifold $W^s(v)$ in $\mathrm{Int}M$;
(3) the restriction of X to ∂M is a Morse flow (the stable and unstable manifolds have a transversal intersection).

We consider Morse flows with singularities on ∂M. There are 6 types of singularities which are determined by their indices.

The pair (p, q) is called the index of a singular point where $p+q$ is equal to dimension of stable manifold of X and p is the dimension of the flow restricted to the boundary. In this case $p = 0, 1$ or 2 and $q = 0$ or 1. For example, a source has index (0,0) and a sink has index (2,1).

The surface F is the closure of intersection of Int M with boundary of a regular neighborhood for the union of the 1-dimensional stable manifolds.

Arcs and circles $\{u, U, v, V\}$ on F are intersections of unstable manifolds for singular points of index (1,0), (0,1),(1,1),(2,0) and the surface F.

The set (F, u, U, v, V) consisting of a surface with boundary, a set of circles and arcs embedded in it as described above is called a Morse flow diagram.

Theorem 1. *Two Morse-Smale flows on 3-manifold with a boundary are topologically trajectory equivalent if and only if their diagrams are homeomorphic.*

Morse flow diagrams have the following properties:

(1) $U_i, V_i \subset \mathrm{Int}M$, $\mathrm{Int}u_i, \mathrm{Int}v_i \subset \mathrm{Int}M$, $\partial u_i, \partial v_i \subset \partial M$;
(2) $\partial U_i \partial \cup_i u_i, \partial V_i \partial \cup_i v_i$;
(3) $U_i \cap U_j = \emptyset$ if $i \neq j$, $\quad u_i \cap u_j = \emptyset$ if $i \neq j$, $\quad V_i \cap V_j = \emptyset$ if $i \neq j$, $\quad v_i \cap v_j = \emptyset$ if $i \neq j$, $u_i \cap U_j = \emptyset$, $\quad v_i \cap V_j = \emptyset$, $\quad \partial u_i \cap \partial v_j = \emptyset$.
(4) U_k is a closed curve or it belongs to a left-hand turn cycle which consists of U_i and u_j; the similar property holds true for V_k.
(5) if we cut F along u_i and do spherical surgeries by U-cycles then we get a union of 2-disks.

Theorem 2. *If a surface F with 4 sets of curves has the properties 1-5, then it is a diagram of a Morse flow.*

References

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