Weak Separation Condition coincides Finite Type Condition

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Geometric and Topological properties of self similar sets are well understood when the associated *Iterated Function System (IFS)* satisfies *Open Set Condition (OSC)*[1]. OSC fails if IFS is said to have overlaps. Lau and Ngai in [2] introduced *Weak separation property (WSP)* which allows limited overlaps of copies and is less restrictive than OSC. This separation property was extensively studied by Zerner [3]. The notion of separation allows to explicitly calculate the Hausdorff dimension of self-similar sets. But it can be still challenging. Another notion of separation termed as *Finite Type Condition (FTC)* introduced in [3] by , which enhances the domain of self similar sets. The question of equivalence was first raised in [4], they proved that it was not true in general for $d > 1$ IFS in $\mathbb{R}^n$. We characterise Weak separation property in terms of neighbourhood sets.

Notations and Definitions

Let $\mathcal{S} = \{S_1, ..., S_m\}$ be system of contracting maps on $\mathbb{R}$: $|S_i(x) - S_j(y)| = r_i|x-y|$, where $1 > r_i > 0$. Then $\exists$ unique compact set K such that $K = \bigcup_{i=1}^m S_i(K)$. K is called *self similar set* associated to IFS $\mathcal{S}$. OSC states that there is an open set $O \neq \phi$ such that $S_i(O) \subseteq O$ and $S_i(O) \cap S_j(O) = \phi$ for $i, j \in \{1, ..., m\}$. Let $I = \{1, ..., m\}$ be finite set of symbols and and $\mathbf{i}, \mathbf{j}$ be words from $I^* = \bigcup\{I^n : n = 1, 2, ...\}$. From [5] consider the following :

$$F = \{S_{\mathbf{i}}^{-1} \cdot S_{\mathbf{j}} \;\; ; \;\; \mathbf{i}, \mathbf{j} \in I^*\}$$

subset of topological group $\mathcal{G}$ of all similarities on $\mathbb{R}$. From [6] given any $a > 0$, let

$$I_a = \{\mathbf{i} = i_1 i_2 \ldots i_n \in I^*; |r_{\mathbf{i}}| < a \leq |r_{i_1 i_2 \ldots i_{n-1}}|\}$$

IFS satisfy weak separation property if there are $x \in \mathbb{R}$ and integer $l \in \mathbb{N}$ such that for any $a > 0$ and finite word σ, every closed ball with radius a, contains atmost l distinct elements of type $S_{\mathbf{i}}((S_\sigma(x))$ for $\mathbf{i} \in I_a$.

Definition : 1 - The notion of *Neighbourhood Sets* defined in [7] is very helpful to study finite type condition. For $\alpha \in \mathbb{Z}$, let $h_1, ..., h_{m_\alpha}$ be elements of set $\{S_{\mathbf{i}}(0), S_{\mathbf{i}}(1) : \mathbf{i} \in I_a\}$. Let $\mathcal{F}$ be union of all possible net intervals such that

$$\mathcal{F}_\alpha = \{[h_i, h_{i+1}] : 1 \leq i \leq m_\alpha\}$$

Suppose $\triangle \in \mathcal{F}$ and denote contraction map $T_\triangle(x) = rx + c$ where $r > 0$ such that $T_\triangle([0,1]) = \triangle$. Similarity $T(x) = Lx + c$ is neighbourhood set of $\mathcal{F}_a$ if $\exists$ $\mathbf{i} \in I_a$ such that

$$\triangle \subseteq S_{\mathbf{i}}([0,1]) \quad \text{and} \quad T = T_\triangle^{-1} \circ S_{\mathbf{i}}$$

Definition : 2 IFS satisfies *finite neighbourhood condition* if it has finite neighbourhood set.

The main result follows the following lemma.

Lemma : 1 Presume that finite neighbourhood condition holds for system $\mathcal{S}$. Then $\exists$ $l > 0$ such that any $1 \geq a > 0$, $\mathbf{i}, \mathbf{j} \in I_a$ and $p, q \in \{0, 1\}$ either

$$S_{\mathbf{i}}(p) = S_{\mathbf{j}}(q) \quad \text{or} \quad |S_{\mathbf{i}}(p) - S_{\mathbf{j}}(q)| \geq la$$

Lemma : 2 Suppose that $K = [0,1]$ is self similar set of IFS $\mathcal{S}$ and WSP holds. For $\delta > 0$, $\exists$ a finite set $\mathcal{N}_\delta$ so that for any $a > 0$ and $\mathbf{i}, \mathbf{j} \in I_a$ either

$$\mu(S_{\mathbf{i}}([0,1]) \cap S_{\mathbf{j}}([0,1])) < \delta a \quad \text{or} \quad S_{\mathbf{i}}^{-1} \circ S_{\mathbf{j}} \in \mathcal{N}_\delta$$

Theorem : Let $K = [0, 1]$ be the self similar set associated to system of contraction maps $\mathcal{S}$ such that weak separation property holds. Then finite neighbourhood condition is satisfied for $\mathcal{S}$.

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