IW contractions and their generalizations

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Contractions of Lie algebras are a kind of limit processes between orbits of such algebras. In 1953, Inönü and Wigner studied special contractions of Lie algebras as a part of broader study of contractions of Lie groups and their representations. These contractions were generalized by Doebner and Melsheimer in 1967. A rigorous general definition of contractions of Lie algebras was given by Saletan in 1961. He also studied contraction whose matrices are first-order polynomials with respect to contraction parameters. Since then, a number of conjectures about various ways of realizing contractions of Lie algebras had accumulated in the literature.

Definition 1. Let V be an n-dimensional vector space over $\mathbb{F}=\mathbb{C}$ or $\mathbb{F}=\mathbb{R}$, $n<\infty$, and let $\mathcal{L}_n=\mathcal{L}_n(\mathbb{F})$ denote the set of all possible Lie brackets on V. Given $\mu\in\mathcal{L}_n$ and $U\in\mathrm{C}\big((0,1],\mathrm{GL}(V)\big)$, define the family of $\mu_\varepsilon\in\mathcal{L}_n$, $\varepsilon\in(0,1]$, by $\mu_\varepsilon(x,y):={U_\varepsilon}^{-1}\mu(U_\varepsilon x,U_\varepsilon y)\ \forall\, x,y\in V$. If for any $x,y\in V$ there exists the limit $\lim_{\varepsilon\to+0}\mu_\varepsilon(x,y)=:\mu_0(x,y)$, then $\mathfrak{g}_0=(V,\mu_0)$ is a well-defined Lie algebra and is called a *contraction* of the Lie algebra $\mathfrak{g}=(V,\mu)$. The procedure $\mathfrak{g}\to\mathfrak{g}_0$ providing $\mathfrak{g}_0$ from $\mathfrak{g}$ is also called a *contraction*. If a basis of V is fixed, the parameter matrix $U_\varepsilon=U(\varepsilon)$, $\varepsilon\in(0,1]$, is called the *contraction matrix* of the contraction $\mathfrak{g}\to\mathfrak{g}_0$.

Definition 2. The contraction $\mathfrak{g}\to\mathfrak{g}_0$ is called *a Inönü–Wigner (IW) contraction* if its matrix U_ε can be represented in the form $U_\varepsilon=AW_\varepsilon P$, where the matrices A and P are nonsingular and constant (i.e., they do not depend on ε) and $W_\varepsilon=\mathrm{diag}(\varepsilon^{\alpha_1},\dots,\varepsilon^{\alpha_n})$ for some $\alpha_1,\dots,\alpha_n\in\mathbb{R}$. The n-tuple of exponents $(\alpha_1,\dots,\alpha_n)$ is called the *signature* of the generalized IW-contraction $\mathfrak{g}\to\mathfrak{g}_0$. A *simple IW-contraction* is a generalized IW-contraction with signature consisting of zeros and ones.

The following assertion, which stood as a conjecture for a long time, was proved in [5].

Theorem 3. *Any generalized IW-contraction is equivalent to a generalized IW-contraction with an integer signature (and the same associated constant matrices).*

One of these conjectures was that any contraction of Lie algebras can be realized as a generalized IW-contraction. This is true for contractions between three-dimensional real or complex Lie algebras. Consider four-dimensional real Lie algebras defined, up to antisymmetry of Lie bracket, by the following nonzero commutation relations:

$$
\begin{aligned}
2A_{2.1}\colon &\quad [e_1,e_2]=e_1,\ [e_3,e_4]=e_3;\\
A_1\oplus A_{3.2}\colon &\quad [e_2,e_4]=e_2,\ [e_3,e_4]=e_2+e_3;\\
A_{4.1}\colon &\quad [e_2,e_4]=e_1,\ [e_3,e_4]=e_2;\\
A_{4.10}\colon &\quad [e_1,e_3]=e_1,\ [e_2,e_3]=e_2,\ [e_1,e_4]=-e_2,\ [e_2,e_4]=e_1.
\end{aligned}
$$

Hereafter we use the Mubarakzyanov's nomenclature for low-dimensional Lie algebras, and $\mathfrak{g}_{\dots}$ denotes the complexification of the algebra $A_{\dots}$. All contractions of four-dimensional real Lie algebras were realized in [1, 2] via generalized IW-contractions except two contractions, $2A_{2.1}\to A_1\oplus A_{3.2}$ and $A_{4.10}\to A_1\oplus A_{3.2}$. Since the complexifications of the algebras $2A_{2.1}$ and $A_{4.10}$ are isomorphic, there was only one exception for the complex case, $2\mathfrak{g}_{2.1}\to\mathfrak{g}_1\oplus\mathfrak{g}_{3.2}$.

Theorem 4 ([4])**.** *(i) There exists a unique contraction between four-dimensional complex Lie algebras, $2\mathfrak{g}_{2.1}\to\mathfrak{g}_1\oplus\mathfrak{g}_{3.2}$, that is not equivalent to a generalized IW-contraction.*

(ii) Precisely two contractions between four-dimensional real Lie algebras, $2A_{2.1} \to A_1 \oplus A_{3.2}$ and $A_{4.10} \to A_1 \oplus A_{3.2}$, cannot be realized as generalized IW-contractions.

Combining the results of [1, 2, 4] also yields the following assertion.

Theorem 5 ([4]). *Any generalized IW-contraction between four-dimensional complex (resp. real) Lie algebras is equivalent to one with parameter exponents in $\{0,1,2,3\}$. The exponents in $\{0,1,2\}$ suffice for all such contractions except $2A_{2.1} \to A_{4.1}$, $A_{4.10} \to A_{4.1}$ and $\mathrm{so}(3) \oplus A_1 \to A_{4.1}$ in the real case and $2\mathfrak{g}_{2.1} \to \mathfrak{g}_{4.1}$ in the complex case, where the minimal tuple of exponents is $(3,2,1,1)$.*

Definition 6. The contraction $\mathfrak{g} \to \mathfrak{g}_0$ is called *diagonal* if its matrix U_ε can be represented in the form $U_\varepsilon = AW_\varepsilon P$, where A and P are constant nonsingular matrices and $W_\varepsilon = \mathrm{diag}(f_1(\varepsilon), \dots, f_n(\varepsilon))$ for some continuous functions $f_i\colon (0,1] \to \mathbb{F}\backslash\{0\}$.

Theorem 7 ([5]). *Any diagonal contraction is equivalent to a generalized IW-contraction with an integer signature.*

Consider the n-dimensional ($n \geqslant 5$) solvable real Lie algebras $\mathfrak{a} := A_{5.38} \oplus (n-5)A_1$ and $\mathfrak{a}_0 := A_{2.1} \oplus A_{2.1} \oplus (n-4)A_1$ whose nonzero commutation relations are exhausted, up to antisymmetry of Lie bracket, by the following:

$$\mathfrak{a}\colon\quad [e_1,e_3]=e_3,\quad [e_2,e_4]=e_4,\quad [e_1,e_2]=e_5,\qquad \mathfrak{a}_0\colon\quad [e_1,e_3]=e_3,\quad [e_2,e_4]=e_4.$$

Theorem 8 ([3]). *The Euclidean norm of any contraction matrix that realizes the contraction of the algebra $\mathfrak{a}$ to the algebra $\mathfrak{a}_0$ approaches infinity at the limit point. The same is true for the complex counterpart of this contraction.*

Definition 9. A realization of a contraction with a matrix-function that is linear in the contraction parameter is called a *Saletan (linear) contraction*.

Theorem 10 ([6]). *Up replacing the algebras $\mathfrak{g}$ and $\mathfrak{g}_0$ with isomorphic ones, every Saletan contraction $\mathfrak{g} \to \mathfrak{g}_0$ is realized by a matrix of the canonical form*

$$E^{n_0} \oplus J_\varepsilon^{n_1} \oplus \dots \oplus J_\varepsilon^{n_s}, \qquad \textit{or, equivalently,} \qquad E^{n_0} \oplus J_0^{n_1} \oplus \dots \oplus J_0^{n_s} + \varepsilon E^n,$$

where $n_0 + \dots + n_s = n$, E^m is the $m \times m$ identity matrix, and J_λ^m denotes the $m \times m$ Jordan block with an eigenvalue λ.

Hence any Saletan contraction can be realized by a matrix of the form $AS_\varepsilon B$, where A and B are constant nonsingular matrices and the matrix-valued function S_ε is in the above canonical form. The tuple $(n_0; n_1, \dots, n_s)$, where $n_1, \dots, n_s$ constitute a partition of the dimension $n - n_0$ of the Fitting null component relative to U_0 and $n_0 \in \{0, \dots, n\}$, is called the *signature* of this Saletan contraction.

References

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