On non-Hausdorff manifolds of dimension 1

Sergiy Maksymenko
(Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine)
E-mail: maks@imath.kiev.ua

Eugene Polulyakh
(Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine)
E-mail: polulyah@imath.kiev.ua

Consider a T_1 topological space Y that is locally homeomorphic with $[0, 1)$. In other words, Y is a one-dimensional non-Hausdorff but T_1 manifold.

As usual, a point $y \in Y$ is *internal*, if it has an open neighborhood homeomorphic with $(0, 1)$. Otherwise, y has an open neighborhood homeomorphic with $[0, 1)$ and is called a *boundary* point. The set of all internal and boundary points will be denoted by $\mathrm{Int}Y$ and ∂Y respectively.

For a point $z \in Y$ define its *Hausdorff closure*, $\mathrm{hcl}(z)$, to be the intersection of closures of all neighbourhoods of z, that is

$$\mathrm{hcl}(z) := \bigcap_{V \text{ is a neighbourhood of } z} \overline{V}.$$

Evidently, $z \in \mathrm{hcl}(z)$. We say that z is *special* whenever $\mathrm{hcl}(z) \setminus z \neq \varnothing$. Denote by $\mathcal{V}$ the set of all special points of Y.

Let $\mathcal{H}(Y)$ be the group of homeomorphisms of Y endowed with compact open topology, and $\mathcal{H}_{\mathrm{id}}(Y)$ be the identity path component of $\mathcal{H}(Y)$, so it is a normal subgroup consisting of homeomorphisms isotopic to the identity. The following statement gives a characterization of $\mathcal{H}_{\mathrm{id}}(Y)$ under assumption that the set $\mathcal{V}$ of special points of Y is locally finite.

Theorem 1. *Let Y be a second-countable T_1 topological space being locally homeomorphic with $[0, 1)$ and such that the set $\mathcal{V}$ of its special points is locally finite. Let also $k \in \mathcal{H}(Y)$. Then $k \in \mathcal{H}_{\mathrm{id}}(Y)$ if and only if the following two conditions hold:*

(1) *k fixes each special point of Y;*
(2) *k preserves orientation of each connected component e of $Y \setminus \mathcal{V}$.*