Properties of quasisymmetric mappings to preserve the structures of spaces

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The class of quasisymmetric mappings on the real axis was first introduced by A. Beurling and L. V. Ahlfors [1]. Later P. Tukia and J. Väisälä [2] considered these mappings between general metric spaces. See, e.g., [3] for an overview of the results in this direction. In our work we generalize the concept of quasisymmetric mappings to the case of general semimetric spaces. We establish conditions under which the image $f(X)$ of a semimetric space X with the triangle function Φ_1 under η-quasisymmetric embedding f is a semimetric space with another triangle function Φ_2. Condition under which f preserves a Ptolemy inequality is also found as well as condition under which f preserves a relation "to lie between" imposed on three different points of the space.

Let X be a nonempty set. Recall that a mapping $d\colon X \times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$ is a *metric* if for all $x, y, z \in X$ the following axioms hold: (i) $(d(x,y) = 0) \Leftrightarrow (x = y)$, (ii) $d(x,y) = d(y,x)$, (iii) $d(x,y) \leqslant d(x,z) + d(z,y)$. The pair (X,d) is called a *metric space*. If only axioms (i) and (ii) hold then the pair (X,d) is called a *semimetric space*.

Definition 1. Let (X,d), (Y,ρ) be semimetric spaces. We shall say that an embedding $f\colon X \to Y$ is *η-quasisymmetric* if there is a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ so that

$$d(x,a) \leqslant t d(x,b) \text{ implies } \rho(f(x), f(a)) \leqslant \eta(t) \rho(f(x), f(b))$$

for all triples a, b, x of points in X and for all $t > 0$.

A definition of a triangle function was introduced by M. Bessenyei and Z. Páles in [4].

Definition 2. Consider a semimetric space (X,d). We say that $\Phi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a *triangle function* for d if Φ is symmetric and monotone increasing in both of its arguments, satisfies $\Phi(0,0) = 0$ and, for all $x, y, z \in X$, the following generalized triangle inequality holds:

$$d(x,y) \leqslant \Phi(d(x,z), d(y,z)).$$

The most important triangle functions $\Phi(u,v)$ which generate well-known types of metrics and their generalizations are $u + v$ (metric), $K(u+v)$ (b-metric with $K \geqslant 1$), $\max\{u,v\}$ (ultrametric).

Proposition 3. *Let (X,d) be a semimetric space with the triangle function Φ_1, (Y,ρ) be a semimetric space and let $f\colon X \to Y$ be a surjective η-quasisymmetric embedding. Suppose that the following conditions hold for Φ_1 and for some function $\Phi_2\colon \mathbb{R}_+^2 \to \mathbb{R}_+$:*

(i) *Φ_2 is symmetric, monotone increasing in both of its arguments and satisfies $\Phi(0,0) = 0$,*
(ii) *$\lambda\Phi_1(x,y) \leqslant \Phi_1(\lambda x, \lambda y)$ and $\Phi_2(\lambda x, \lambda y) \leqslant \lambda \Phi_2(x,y)$ for every $\lambda > 0$,*
(iii) *For every $t_1, t_2 \in \mathbb{R}_+ \setminus \{0\}$ the inequality*

$$1 \leqslant \Phi_1\left(\frac{1}{t_1}, \frac{1}{t_2}\right) \text{ implies } 1 \leqslant \Phi_2\left(\frac{1}{\eta(t_1)}, \frac{1}{\eta(t_2)}\right). \tag{1}$$

Then Φ_2 is a triangle function for the space (Y,ρ).

In what follows under Ptolemaic spaces we understand semimetric spaces (X,d) for which the well-known Ptolemy inequality

$$d(x,z)d(t,y) \leqslant d(x,y)d(t,z) + d(x,t)d(y,z)$$

holds. Note that this inequality does not imply the standard triangle inequality in (X,d).

Proposition 4. *Let (X,d) be a Ptolemaic space, (Y,ρ) be a semimetric space and let $f\colon X \to Y$ be a surjective η-quasisymmetric embedding. If for every $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ the inequality*

$$t_1t_2t_3t_4 \leqslant t_1t_2 + t_3t_4 \ \text{ implies } \ \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) \leqslant \eta(t_1)\eta(t_2) + \eta(t_3)\eta(t_4), \tag{2}$$

then (Y,ρ) is also Ptolemaic.

Let (X,d) be a semimetric space and let x, y, z be different points from X. We shall say that the point y lies between x and z if the equality $d(x,z) = d(x,y) + d(y,z)$ holds. K. Menger [5] seems to be the first who formulated the concept of "metric betweenness" for general metric spaces.

Theorem 5. *Let (X,d), (Y,ρ) be semimetric spaces and let $f\colon X \to Y$ be η-quasisymmetric embedding. If the homeomorphism η has the form*

$$\eta(t) = \begin{cases} \frac{1}{2} + \Psi_1(t, 1-t), & t \in [0,1], \\ \frac{1}{\frac{1}{2} + \Psi_2\left(\frac{1}{t}, 1 - \frac{1}{t}\right)}, & t \in [1+\infty), \end{cases} \tag{3}$$

where Ψ_1, Ψ_2 are some continuous, antisymmetric, strictly increasing by the first variables, defined on $[0,1]\times[0,1]$ functions of two variables such that $\Psi_1(1,0) = \Psi_2(1,0) = 1/2$, then f preserves metric betweenness.

References

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