Centralizers of elements in Lie algebras of vector fields

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Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $A = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring over $\mathbb{K}$. A $\mathbb{K}$-derivation D of A is a $\mathbb{K}$-linear mapping $D\colon A \to A$ that satisfies the rule: $D(ab) = D(a)b + aD(b)$ for all $a,\ b \in A$. If $\mathbb{K} = \mathbb{R}$ then every derivation D on $\mathbb{R}[x_1, \ldots, x_n]$ can be considered as a vector field on $\mathbb{R}^n$ with polynomial coefficients. The vector space $W_n(\mathbb{K})$ (over the field $\mathbb{K}$)of all $\mathbb{K}$-derivations (or vector fields) on the polynomial ring A is a Lie algebra over $\mathbb{K}$. Any derivation $D \in W_n(\mathbb{K})$ can be uniquely extended on the field $R = \mathbb{K}(x_1, \ldots, x_n)$ of rational functions in n variables by the rule: $D(a/b) = (D(a)b - aD(b))/b^2$, the vector space $\widetilde{W_n}(\mathbb{K})$ of all derivations on R is also a Lie algebra, it is in fact the Lie algebra of all vector fields with rational coefficients on $\mathbb{K}^n$.

Recall that for a given Lie algebra L and its element $x \in L$ the set $C_L(x) = \{y \in L :\ [x, y] = 0\}$ is called the centralizer of x in L. The centralizer $C_L(x)$ is a subalgebra of the Lie algebra $W_n(\mathbb{K})$ containing the element x. The structure of centralizers of polynomial derivations is of significant importance due to applications in differential equations and geometry (see, for example [1], [2]).

Let p and q be algebraically independent irreducible polynomials from the ring A. A polynomial $f \in A$ will be called p-q-free if f is not divisible by any homogeneous polynomial in p and q of positive degree. One can write every polynomial $g \in A$ in the form $g_0 g_1$, where g_0 is a p-q-free polynomial and $g_1 = h(p, q)$ for some homogeneous polynomial $h(s, t) \in \mathbb{K}[s, t]$. The (total) degree of h in s, t will be called the p-q-degree of g and denoted by $\deg_{p-q} g$. The following result gives a characterization of a centralizer of a polynomial derivation if its field of constants (in the field of rational functions) satisfies certain restrictions:

Theorem 1. *Let $D_1 \in W_n(\mathbb{K})$ be such a derivation that its field of constants $\mathrm{Ker} D_1$ in the field of rational functions $\mathbb{K}(x_1, \ldots, x_n)$ is of transcendence degree one and contains no nonconstant polynomials. Then $\mathrm{Ker} D_1 = \mathbb{K}(\frac{p}{q})$ for some irreducible and algebraically independent polynomials $p, q \in \mathbb{K}[x_1, \ldots, x_n]$ and $D_1 = hf(p, q)D_0$ for some homogeneous polynomial f in the variables p, q, p-q-free polynomial h and irreducible derivation D_0. Further, the centralizer $C = C_{W_n}(D)$ is one of the following Lie algebras: (1) $C = \mathbb{K}[p, q]_m hD_0$, where $\mathbb{K}[p, q]_m$ is the linear space of homogeneous polynomials in p, q and $m = \deg_{p\text{-}q} f$, (2) $C = (\mathbb{K}(\frac{p}{q})D_1 + \cdots + \mathbb{K}(\frac{p}{q})D_k) \cap W_n(\mathbb{K})$ for some linearly independent with D_1 derivations $D_2, \ldots, D_k \in C$ over the field $\mathbb{K}(x_1, \ldots, x_n)$.*

Moreover, C is of finite dimension over the field $\mathbb{K}$.

References

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