Homotopic Nerve Complexes with Free Group Presentations

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This paper introduces homotopic nerve complexes in a planar Whitehead CW space [6, §4-5] and their Rotman free group presentations [4, §11,p.239]. A CW complex in a space K is a closure-finite cell complex that is Hausdorff (union of disjoint cells), satifying the containment property (closure of every cell complex is in K) and intersection property (common parts of cell complexes in K are also in K). A complex K is locally finite, i.e., every point $p \in K$ is a member of some finite subcomplex of K and every complex has a finite number of faces [5, §5.2,p.65]. A planar CW complex K is a collection of 0-cells (vertexes), 1-cells (edges) and 2-cells (filled triangles). Collections of planar cells attached to each other are sub-complexes in K.

Definition 1. 1-Cycle. A 1-cycle $\mathrm{cyc}E$ in a CW space K is a collection of path-connected vertexes on 1-cells (edges) attached to each other and with no end vertex. ■

The edges in a 1-cycle cell complex in a CW space are replaced by homotopic maps to obtain a homotopic cycle.

Definition 2. Homotopic-Cycle. A **homotopic cycle** E (denoted by $\mathrm{cyc}E$) is defined to be $\{h_i\}_{i=1}^{n}$, a set of n paths in a space X, where $h_1(0) = h_n(1)$ and the initial point of h_{i+1} is the terminal point of h_i for $2 \leq i \leq n-1$, i.e., $h_i(0) = h_{(i-1)}(1)$. Each path is a mapping $h : [0,1] \to X$ and $h_i(0)$ is a vertex in a finite set of cycle vertices. A reverse path $\bar{h}_i(t) := h_i(t-1)$ gives us an inverse map, so that

$$h_i(0) - \bar{h}_i(1) = h_i(0) - h_i(1-1) = h_i(0) - h_i(0) = 0.$$

In cycle $\mathrm{cyc}E$, every vertex v_i is reachable by k maps from a distinguished vertex $h_1(0) = v_0$, *i.e.*,

$$\begin{aligned} kv_0 &:= h_1(0) + \cdots + h_{k+1}(0) \\ &\overbrace{:= h_1 \to \cdots \to h_{k+1}}^{\textit{i.e., } k \textbf{ maps to reach } h_{k+1}(0) \textbf{ from } h_1(0)}. \end{aligned}$$

Here, $+$ represents a move from one vertex to another one in the cycle, which translates to a homotopic path between vertices. ■

Definition 3. Nerve Complex. A nerve complex $\mathrm{Nrv}E$ in a space X is a collection of nonempty cell complexes with nonvoid intersection. ■

Theorem 4. *A pair of pair of 1-cycles with a common vertex in a CW space is a nerve complex.*

Theorem 5. *Every collection of homotopic cycles with a common vertex in a CW space is a homotopic nerve complex.*

Lemma 6. *Every vertex in the triangulation of the vertices in a CW space is the nucleus of an Alexandroff-Hopf nerve complex* [1, §4.2.11, p. 161].

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Theorem 7. *A CW space containing n triangulated vertexes contains n Alexandroff-Hopf nerve complexes.*

Remark 8. A finite group G is free, provided every element $x \in G$ is a linear combination of its basis elements (called generators) [2, §1.4, p. 21]. We write $\mathcal{B}$ to denote a nonempty basis set of generators $\left\{g_1, \dots, g_{|\mathcal{B}|}\right\}$ and $G(\mathcal{B}, +)$ to denote the free group with binary operation $+$. ■

Definition 9. Rotman Presentation[4, p.239] Let $X = \{g_1, \dots\}, \triangle = \{v = \sum k g_i, v, g_i \in X\}$ be a set of generators of members of a nonempty set X and set of relations between members of X and the generators in X. A mapping of the form $\{X, \triangle\} \to G$, a free group, is called a presentation. ■

Definition 10. Let 2^K be the collection of cell complexes in a CW space K, $E \in 2^K$, basis $\mathcal{B} \in G$, k_i the i^{th} integer coeficient in a linear combination $\sum\limits_{i,j} k_i g_j$ of generating elements $g_j \in \mathcal{B}$. A free group G **presentation** of E is a continuous map $f : 2^K \to 2^K$ defined by

$$f(E) = \left\{ v := \sum_{i,j} k_i g_j : v \in E, g_j \in \mathcal{B}, k_i \in \mathbb{Z} \right\}$$
$$= \overbrace{\boldsymbol{G}(\left\{\boldsymbol{g_1, \dots, g_{|\mathcal{B}|}}\right\}, +)}^{\boldsymbol{E} \mapsto \textbf{free group } \boldsymbol{G}}. \quad ■$$

Lemma 11. [3, §4, p. 10] *Every homotopic cycle in a space X has a free group presentation.*

Here are two main results.

Theorem 12. *Every homotopic cycle in a CW space has a free group presentation.*

Theorem 13. *Every homotopic nerve in a CW space has a free group presentation.*

Remark 14. An application of nerve complexes is given in terms of the approximation of video frame shapes. ■

References

1. P. Alexandroff and H. Hopf, *Topologie. Band i*, Springer, Berlin, 1935, Zbl 13, 79; reprinted Chelsea Publishing Co., Bronx, N. Y., 1972. iii+637 pp., MR0345087.
2. J.R. Munkres, *Elements of algebraic topology, 2^{nd} ed.*, Perseus Publishing, Cambridge, MA, 1984, ix + 484 pp., ISBN: 0-201-04586-9, MR0755006.
3. J.F. Peters, *Amiable and almost amiable fixed sets. extension of the brouwer fixed point theorem*, GLASNIK MATEMATIČKI **35** (2020), no. 75, 1–17, arXiv.2008.07584v4.
4. J.J. Rotman, *The theory of groups. An introduction.*, Allyn and Bacon, Inc., Boston, Mass., 1965, xiii+305 pp., MR0204499, reviewed by R.C. Lyndon.
5. R.M. Switzer, *Algebraic topology – homology and homotopy*, Springer, Berlin, 2002, xii+526 pp., Zbl 1003.55002.
6. J.H.C. Whitehead, *Combinatorial homotopy. I*, Bulletin of the American Mathematical Society **55** (1949), no. 3, 213–245, type 2 complexes:MR0030759,3D complexes:MR0005352.