On representations of q_{ij}-commuting isometries

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C^*-algebras generated by isometries have been studied by various authors. Among the most relevant examples we mention Toeplitz algebras, Cuntz algebras, and their deformations. These examples belong to the class of $*$-algebras with Wick ordering [1].

Recall that the Cuntz-Toeplitz algebra $\mathcal{O}_d^0$ is a unital C^*-algebra generated by elements s_j, $j = 1, \ldots, d$, which satisfy relations

$$s_j s_k = \delta_{jk} I, \quad j, k = 1, \ldots, d.$$

In this paper, we consider representations of C^*-algebra W_d generated by elements s_j, $j = 1, \ldots, d$, satisfying relations

$$s_i^* s_i = I, \quad s_i^* s_j = q_{ij} s_j s_i^*, \quad |q_{ij}| < 1,\ q_{ij} = \bar{q}_{ji}, \qquad 1 \le i \ne j \le d. \tag{1}$$

One can see that for $q_{ij} = 0$, $i \ne j$, this algebra is $\mathcal{O}_d^0$. It was conjectured in [2] that, in particular, for $|q_{ij}| < 1$, $i \ne j$, the corresponding C^*-algebra is isomorphic to $\mathcal{O}_d^0$, however, the proof is known for the cases $d = 2$ [3] or $|q_{ij}| < \sqrt{2} - 1$ [2] only. While the representations of the Cuntz-Toeplitz algebras were studied in detail in a number of papers, for other Wick algebras, including W_d, only the Fock representation [1] is known. Therefore, constructing representations of "deformed" relations (1) can give a hint for a construction of the isomorphism between W_d and $\mathcal{O}_d^0$ in a general case.

We start with some notations. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \{1, \ldots, d\}^m$ be a finite multiindex of length m, $|\alpha| = m$, let $\Lambda_m = \{1, \ldots, d\}^m$ be the set of all finite multiindices of length m, $\Lambda_0 = \emptyset$, and let $\Lambda^0 = \cup_{m=0}^{\infty} \Lambda_m$ be the set of all finite multiindices of arbitrary length. Also, we will use the set $\Lambda = \{1, \ldots, d\}^\infty$ of all infinite multiindices. For each finite multiindex $\alpha = (\alpha_1, \ldots, \alpha_m) \in \Lambda^0$ we use notation $s_\alpha = s_{\alpha_1} \ldots s_{\alpha_m}$. For a finite multiindex we use standard mappings:

$$\Lambda_m \ni \alpha = (\alpha_1, \ldots, \alpha_m) \mapsto \sigma(\alpha) = (\alpha_2, \ldots, \alpha_m) \in \Lambda_{m-1},$$
$$\Lambda_m \ni \alpha = (\alpha_1, \ldots, \alpha_m) \mapsto \sigma_j(\alpha) = (j, \alpha_2, \ldots, \alpha_m) \in \Lambda_{m+1}, \quad j = 1, \ldots, d.$$

The same mappings can be obviously defined for an infinite multiindex $\alpha \in \Lambda$.

If $\alpha \in \Lambda^0$ does not contain j, then (1) implies

$$s_j^* s_\alpha = q(j, \alpha) s_\alpha s_j^*, \quad q(j, \alpha) = q_{j\alpha_1} \ldots q_{j\alpha_m}.$$

If α contains j, then α can be represented as $\alpha = (\alpha' j \alpha'')$, where α' does not contain j, then

$$s_j^* s_\alpha = q(j, \alpha') s_{\alpha'} s_{\alpha''} = q(j, \alpha) s_{\alpha \setminus j}$$

(here and below, we denote by $\alpha \setminus j = (\alpha' \alpha'')$ multiindex obtained from α by removing the first occurrence of j, and set $q(j, \alpha) = q(j, \alpha')$ for convenience).

For infinite multiindices $\alpha, \beta \in \Lambda$, we define $q(\alpha, \beta)$ as follows. If there exists $\gamma \in \Lambda$, $\alpha', \beta' \in \Lambda_m$, $m \geq 0$, for which

$$\alpha = (\alpha'\gamma), \ \beta = (\beta', \gamma), \quad \alpha' \text{ and } \beta' \text{ coincide up to a permutation,}$$

then we define $q(\alpha, \beta) = q(\alpha', \beta')$, and zero otherwise. It is a straightforward fact that $q(\alpha, \beta)$ is well-defined.

We proceed with introducing an appropriate Hilbert space. We say that infinite multiindices $\alpha, \beta \in \Lambda$ are equivalent, denoted by $\beta \sim \alpha$, if they "have the same tails up to a shift", i.e., there exist numbers m, n, such that $\sigma^m(\alpha) = \sigma^n(\beta)$. Fix an infinite multiindex α and consider a family of vectors $(e_\beta \mid \beta \sim \alpha)$. For these vectors, define

$$(e_\beta, e_\gamma) = q(\beta, \gamma), \tag{2}$$

in particular, $(e_\beta, e_\beta) = 1$.

Proposition 1. *Form* (2) *is well-defined and positive.*

For a fixed $\alpha \in \Lambda$, define a Hilbert space H_α as the closed linear span of vectors $(e_\beta \mid \beta \sim \alpha)$ with respect to the introduced scalar product.

Theorem 2. *1. Operators in* H_α

$$\pi_\alpha(s_j)e_\beta = e_{\sigma_j(\beta)}, \quad \pi_\alpha(s_j^*)e_\beta = \begin{cases} 0, & \beta \text{ does not contain } j, \\ q(j, \beta)e_{\beta \setminus j}, & \text{otherwise,} \end{cases}$$

form well-defined $*$*-representation of the* C^**-algebra* W_d.

2. This representation is irreducible

3. Representations corresponding to multiindices α, α' *are unitary equivalent iff the corresponding Hilbert spaces coincide, i.e.,* $\alpha \sim \alpha'$.

References

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