On m-convexity and m-semiconvexity of sets in Euclidean spaces

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The topological and geometric properties of classes of generally convex sets in multidimensional real Euclidean space $\mathbb{R}^n$, $n \geq 2$, known as m-convex, weakly m-convex, m-semiconvex, and weakly m-semiconvex, $m = 1, 2, \ldots, n-1$, are studied in [1]–[6]. A set of the space $\mathbb{R}^n$ is called ***m*-convex** (***m*-semiconvex**) if for any point of the complement of the set to the whole space there is an m-dimensional plane (half-plane) passing through this point and not intersecting the set. An open set of the space is called ***weakly m*-convex** (***weakly m*-semiconvex**), if for any point of the boundary of the set there exists an m-dimensional plane (half-plane) passing through this point and not intersecting the given set. A closed set of the space is called ***weakly m*-convex** (***weakly m*-semiconvex**) if it is approximated from the outside by a family of open weakly m-convex (weakly m-semiconvex) sets. These notions were proposed by Professor Yuri Zelinskii [1], [2].

Let us denote the classes of m-convex and weakly m-convex sets in $\mathbb{R}^n$, $n \geq 2$, by $\mathbf{C^n_m}$ and $\mathbf{WC^n_m}$, respectively. There are weakly m-convex sets in $\mathbb{R}^n$, $n \geq 2$, $1 \leq m < n$, which are not m-convex, i. e., the class $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$ is not empty for any $m = 1, 2, \ldots, n-1$. The example of an open set of the class $\mathbf{WC^2_1} \setminus \mathbf{C^2_1}$ is constructed in [4]. The examples of open and closed sets of $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ and examples of open sets of $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$, $n \geq 3$, $1 \leq m < n-1$, are constructed in [6]. Moreover, any open or compact set of $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ is necessarily disconnected, but there exist domains of $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$, $n \geq 3$, $1 \leq m < n-1$, which show the following three theorems.

Theorem 1. ([4]) *An open set of the class $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ consists of at least three connected components.*

Theorem 2. ([6]) *A compact set of the class $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ consists of at least three connected components.*

Theorem 3. ([6]) *There exist domains of the class $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$, $n \geq 3$, $1 \leq m < n-1$.*

It is also known the topological classification of open (weakly) $(n-1)$-convex sets in the space $\mathbb{R}^n$ with smooth boundary [1], [4]. Each such a set is convex, or consists of no more than two unbounded connected components, or is given by the Cartesian product $E^1 \times \mathbb{R}^{n-1}$, where E^1 is a subset of $\mathbb{R}$.

Let us denote the classes of m-semiconvex and weakly m-semiconvex sets in $\mathbb{R}^n$, $n \geq 2$, by $\mathbf{S^n_m}$ and $\mathbf{WS^n_m}$, respectively. In [3] it is constructed an example of an open set of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$. It is also conjectured that any open set of $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ consists of at least three components. The latter statement is proved in [4]. There can be also constructed sets of $\mathbf{WS^n_{n-1}} \setminus \mathbf{S^n_{n-1}}$ and the example of domains of $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$, $n \geq 3$, $1 \leq m < n-1$, similar to the domains of $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$. The following theorem shows the impossibility of the topological classification of weakly 1-semiconvex sets with smooth boundary similar to the topological classification of open $(n-1)$-convex and weakly $(n-1)$-convex sets with smooth boundary.

Theorem 4. ([5]) *An open, bounded set of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ with smooth boundary consists of at least four connected components.*

References

[1] Yuri Zelinskii. *Multi-valued mappings in analysis.* Kyiv: Naukova Dumka, 1993. (in Russian)
[2] Yu. B. Zelinskii, I. V. Momot. On (n, m)-convex sets. *Ukr. Mat. Zhurnal*, 53(3): 422–427, 2001. (in Russian)

[3] Yuri Zelinskii. Variations to the problem of "shadow". *Zbirn. Prats Inst. Math. NANU*, 14(1): 163 –170, 2017. (in Ukrainian)

[4] H. K. Dakhil. *The shadows problems and mappings of fixed multiplicity.* Candthesis. Kyiv: Institute of Mthematics of NASU, 2017. (in Ukrainian)

[5] Tetiana Osipchuk. On semiconvexity of open sets with smooth boundary in the plane. *Proceedings of the International Geometry Center*, 12(4): 69–88, 2019.

[6] Tetiana Osipchuk. On topological properties of weakly m-convex sets. *Praci IPMM NANU*, 34: 73–82, 2020. (in Ukrainian)