Reducing singularities of smooth functions to normal forms

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The talk is devoted to the "uniform" reduction of C^∞ smooth functions on 2-dimensional manifolds to a canonical form at the singular points of these functions.

Definition 1. A smooth function $f = f(u_1, u_2)$ has a singularity E_6 at its critical point $P \in \mathbb{R}^2$ if

(i) the first and second differentials $df(P) = 0$, $d^2f(P) = 0$, and the third differential $d^3f(P) \neq 0$ and is a perfect cube;

(ii) there exists a vector $v \in \operatorname{Ker} d^3f(P)$ such that $v^4f \neq 0$ (by v^4f we mean the fourth derivative of f along the tangent vector v at P).

Theorem 2 (Reducing E_6 to normal form)**.** *Let the function $f(u_1, u_2)$ have a singularity E_6 at the critical point P. Then, in some neighborhood of P, there is a local coordinate system in which the point P is the origin, and the function has the normal form*

$$f = f(P) + \tilde{x}^3 \pm \tilde{y}^4.$$

Moreover, this coordinate system can be chosen in such a way that the coordinate change $(u_1, u_2) \to (\tilde{x}, \tilde{y})$ can be expressed in terms of the original function and its partial derivatives of order ≤ 7 using algebraic operations and the operation of taking a proper integral.

Remark 3. In [1], the first part of Theorem 2, i.e. the existence of a coordinate change, was proved using Tougeron's theorem [2]; in view of this, obtaining a formula for the corresponding coordinate change requires solving the Cauchy problem for a system of ordinary differential equations. We construct our coordinate change explicitly, without using Tougeron's theorem.

Our proof of Theorem 2 consists of three steps. At first, we consider a smooth function in two variables of the form $f(u_1, u_2) = u_1^3 + u_2^4 + R(u_1, u_2)$, where the Taylor series of the function $R(u_1, u_2)$ at the origin has zero coefficients at all monomials of the form $u_1^k u_2^l$, $4k + 3l \leq 12$. At second, by the sequence $(u_1, u_2) \to (x, y)$ of substitutions described in [1], we reduce the function f to such a form that the origin leaves fixed and $f^{(k+l)}_{x^k y^l}(0,0) = 0$, where $0 \leq k < 3, 0 \leq l < 4$. At third, using the Taylor series expansion of the function with an integral remainder, we reduce the function to the required normal form by the coordinate change

$$\phi : \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{\tilde{x},\tilde{y}} \qquad \text{with} \quad \tilde{x} = x\sqrt[3]{g(x,y)}, \quad \tilde{y} = y\sqrt[4]{h(x,y)},$$

where $g(x,y) := \sum_{k=0}^{3} \frac{y^k}{k!} f^{(k)}_{y^k}(x,0)/x^3$ and $h(x,y) := \frac{1}{6}\int_0^1 f^{(4)}_{y^4}(x, sy)(1-s)^3\,ds$.

Our next goal is to describe a neighborhood in which the above coordinate change ϕ exists and is regular. To simplify our computations, we will assume that the following is true:

Assumption 4. $f^{(4)}_{y^4}(0,0) = 24$, $f'''_{x^3}(0,0) = 6$.

Theorem 5 (Estimating the radius of a neighborhood where the coordinate change is regular)**.** *Under the hypotheses of Theorem 2 and Assumption 4, let $U_0 = \{(x,y) \mid \max(|x|,|y|) < R_0\}$ be a neighborhood of the origin such that $C_{\alpha\beta} = \sup_{U_0} \left|f^{\alpha+\beta}_{x^\alpha y^\beta}(x,y)\right| \leq M$ for $(\alpha,\beta) \in \{(0,5),(1,4),(3,1),(3,2),(3,3),(4,0),(4,1),(4,2),(4,3)\}$, where $R_0 > 0$, $M \geq 0$. Let's consider the neighborhood $U = \{(x,y) \mid \max(|x|,|y|) < R\}$, where the positive constant R is defined by the formula: $R = \min\{R_0, \frac{2}{M+2}\}$. Let also ϕ be the coordinate change from the proof of Theorem 2 that reduces f to the normal form E_6. Then:*

(a) *the functions $h(x,y)$ and $g(x,y)$ do not change sign in U, that is, the change $\phi|_U$ is well defined and is smooth;*
(b) *at every point $\boldsymbol{x} \in U$, one has $\|\phi'(\boldsymbol{x}) - I\| < C < 1$, where $C = \frac{2}{5}$, i.e., the coordinate change $\phi|_U$ is close to the identity;*
(c) *the coordinate change $\phi|_U$ is injective and regular, i.e., it is an embedding and* $\det|\phi'(\boldsymbol{x})| \neq 0$ *at every point $\boldsymbol{x} \in U$, moreover the image of this embedding contains the disk of radius $(1-C)R$ centred at the origin.*

Remark 6. Our coordinate change $\phi|_U$ from Theorem 2 and Theorem 5 provides a "uniform" reduction of the function f at a singular point of type E_6 to the canonical form $\tilde{x}^3 \pm \tilde{y}^4$ in the sense that the neighbourhood radius and the coordinate change we constructed in this neighbourhood (as well as all partial derivatives of the coordinate change) continuously depend on the function f and its partial derivatives. A uniform reduction of smooth functions near critical points to a canonical form was known earlier for the case of smoothly stable singularities [3]. A uniform reduction of smooth functions to a canonical form by C^k-smooth changes (for finite $k < \infty$) is known for finite type singularities [4] and for topologically stable singularities [5].

To prove Theorem 5, we apply the following lemma to the coordinate transformation ϕ from the proof of Theorem 2. We estimate the norm of the matrix in terms of its elements: $\|A\| \leq \sqrt{\sum a_{ij}^2}$, $A = \{a_{ij}\}_{i,j=1}^n$.

Lemma 7. *Let $\phi : U \to \mathbb{R}^n$ be a smooth mapping, where U is a convex open subset of $\mathbb{R}^n$. Let the differential of ϕ have the form $\phi'(\mathbf{x}) = I + A(\mathbf{x})$, where I is the unit matrix of dimension n, $\|A\| < C$, $0 < C < 1$. Then ϕ is injective and* $\det|\phi'(\mathbf{x})| \neq 0$ *at every point $\mathbf{x} \in U$, i.e., ϕ is a diffeomorphism to its image $\phi(U)$. Moreover, $\langle \phi'(\mathbf{x}), \mathbf{x} \rangle \geq (1-C)|\mathbf{x}|^2$ for every point $\mathbf{x} \in U$.*

From the last assertion of Lemma 7 and [6, Corollary 8.3, Step 1], we conclude that $\phi(U)$ contains the disk of radius $(1-C)R$ centred at the origin. This completes our proof of Theorem 5.

References

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