Two problems in nonholonomic geometry (in quest of a co-worker)

Piotr Mormul
(Institute of Mathematics, University of Warsaw, Banach str. 2, 02-097 Warsaw, Poland)
E-mail: `mormul@mimuw.edu.pl`

Particular distributions living on the stages of the Goursat Monster Tower (GMT) have been the object of studies dating as far back as 1889, then 1896, 1914, 1922, 1978, 1982, 1999, 2001 ... (the very notion GMT is, of course, not that old). From the GMT stage No 8 onwards there exist local numerical invariants (moduli) in the local classification of Goursat distributions (cf. Travaux en cours **62**, Paris (2000), Remarks 3 and 4 on p. 110 and 111).

A. A. Agrachev asked in the year 2000 if the moduli of Goursat structures descend to the level of the nilpotent approximations (NA) – simpler objects retaining some basic properties of initial completely nonholonomic distributions. (The NA's are of central importance in the geometric control theory, motion planning problems, etc.) The author gave two very partial answers to Agrachev's question (in 2006 and 2014, both in the negative). Otherwise, this **problem** remains vastly open.

The talk's proper aim is to survey another **problem** concerning the GMT. The one which asks for all the strongly nilpotent points (or, better: strata) in the stages of the GMT. (In the way of explanation, all points in the GMT are weakly nilpotent in the control theory sense, while only a tiny portion of them is strongly nilpotent.) The conjecture, still unsettled, says that, within the GMT, 'strongly nilpotent' is but a synonym of 'tangential', while all the tangential points are known [already since the mid 2000s] to be strongly nilpotent. So the brunt of this problem boils down to the computation of the NA's at non-tangential points. That little or ... that much.

To just give a non-trivial example, here is a non-tangential stratum RRVRV lying in the 5th stage of the GMT. The associated weights, central in the nonholonomic geometry and analysis, are 1, 1, 2, 3, 5, 7, 11. The NA(RRVRV) computed along the lines of the by-now-classical Bellaiche algorithm is — in certain *adapted* coordinates $z_1, z_2, \dots, z_7$ — spanned by the two vector fields $\partial/\partial z_1$ and

$$\partial/\partial z_2 + z_1\partial/\partial z_3 + z_1z_2\partial/\partial z_4 + z_1z_4\partial/\partial z_5 + z_1z_3z_4\partial/\partial z_6 + \Big(z_1z_3z_6 + \frac{1}{3}z_1z_3^3z_4\Big)\partial/\partial z_7\,.$$

Yet, in this line of research *super*-adapted coordinates are sought, in which a visualisation of a given NA uses as few active variables as possible. In the chosen example such super-coordinates can be derived from the previous ones. The eventual visualisation of the NA(RRVRV) appears to be

$$\Big(\partial/\partial z_1\,, \qquad \partial/\partial z_2 + z_1\partial/\partial z_3 + z_1z_2\partial/\partial z_4 + z_1z_2z_3\partial/\partial z_5 + z_1z_2z_3^2\partial/\partial z_6 + z_1z_2z_3^4\partial/\partial z_7\Big)$$

(it is not possible to visualise NA(RRVRV) in only two active adapted variables; three as in the expansion above is the minimal number). Only having NA(RRVRV) in this utmostly compactified form, it becomes possible to show that the stratum RRVRV is indeed not strongly nilpotent.

The outlined problem is, therefore, pretty much computational. A skilful computer-oriented person is sought in earnest, willing to actively take part in dealing with this challenging problem.