Incomplete spaces of idempotent measures

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The notion of idempotent measure in the idempotent mathematics (i.e., a part of mathematics dealing with idempotent operations on reals) corresponds to that of probability measure in the traditional mathematics [1]. The compact spaces of idempotent measures are intensively investigated by numerous authors. We are going to present some results on topological and categorical properties of the idempotent measures in the noncompact case.

One defines the set of idempotent measures on X as the set of functionals $\mu\colon C(X,[0,1]) \to [0,1]$ satisfying: 1) μ preserves the constants; 2) $\mu(t\phi) = t\mu(\phi)$; 3) $\mu(\phi \vee \psi) = \mu(\phi) \vee \mu(\psi)$.

The set $I(X)$ of idempotent measures is endowed with the weak* topology.

Note that exists a natural (by X) map $\xi_X : I^2(X) \to I(X)$ defined as follows. Given $\phi \in C(X,[0,1])$, let $\bar{\phi}\colon I(X) \to [0,1]$ be the function defined by $\bar{\phi}(\mu) = \mu(\phi)$, $\mu \in I(X)$. Then, for any $M \in I^2(X)$, $\xi_X(M)(\phi) \stackrel{def}{=} M(\bar{\phi})$.

It turns out that the definition of idempotent measure can be formulated in terms of special subsets of $X \times [0,1]$. Namely, we consider subsets $A \subset X \times [0,1]$ of the form: A is closed and saturated; $X \times \{0\} \subset A$; $A \cap (X \times \{1\}) \neq \emptyset$.

To every such set A there corresponds a functional $A : C(X,[0,1]) \to [0,1]$ (we thus keep the same notation) defined by the formula: $A(\phi) = \sup\{t\phi(x) \mid (x,t) \in A\}$.

Let X be a Tychonov space (completely regular space). By βX we denote the maximal compactification (Stone-Čech compactification) of X. Consider the set $I_\beta(X)$ of all subsets A in $X \times [0;1]$ such that :

1) A is closed in $X \times [0,1]$;
2) A is saturated, i.e. $\forall (x,t) \in A\ \forall t', 0 \le t' \le t,\ (x,t') \in A$;
3) $X \times \{0\} \subset A$;
4) the support of A, i.e., the set $supp(A) = \overline{\{x \in X \mid \exists t > 0, (x,t) \in A\}}$ is compact;
5) $\exists x \in X$: $(x,1) \in A$.

We denote by $I_\omega(X)$ the family of all sets $A \in I_\beta(X)$ such that $supp(A)$ is a finite set. Now we define map $\xi : I^2_\omega(X) \to I_\omega(X)$, where ξ

$\xi(A) = \{\ (x,r) \mid \exists\ s,t \in [0,1],\ \alpha \in I_\omega(X)$ such that $r = st, (x,s) \in \alpha,\ (\alpha,t) \in A\ \}$.

Next, we consider the case of metric spaces. For given metric space (X,d), we endow $X \times [0,1]$ by the metric $\hat{d}$, where $\hat{d}((x,t),(x',t')) = d(x,x') \vee |t-t'|$. The space $I_\omega(X)$ is endowed with Hausdorff metric induced by $\hat{d}_H$. We can consider $I_\omega(X)$, as a new metric space with Hausdorff metric d_H. Apply the same operation I_ω to $(I_\omega(X), d_H)$. We obtain a new space $I_\omega(I_\omega(X))$ with (d_{HH}) metric.

Theorem 1. *The map* ξ_X: $(I^2_\omega(X), d_{HH}) \to (I_\omega(X), d_H)$ *is non-expanding.*

This theorem allows us to extend the map ξ_X over the completion $\overline{I}(X)$ of $I_\omega(X)$ (here we assume that X is complete).

References

[1] Litvinov G. L. *The Maslov dequantization, idempotent and tropical mathematics: a brief introduction*, Journal of Mathematical Sciences **140**, #3(2007), 426–444. Also: arXiv:math.GM/0507014