Symplectomorphisms preserving smooth functions on surfaces

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Let M be a compact connected surface and P is a connected one-dimensional manifold without boundary, i.e. either the real line $\mathbb{R}$ or the circle S^1. Denote by $\mathcal{D}(M)$ the group of all smooth (C^∞) diffeomorphisms of M. This group acts from the right on the space $C^\infty(M,P)$ by the following rule: if $h \in \mathcal{D}(M)$ and $f \in C^\infty(M,P)$, then the result of the action of h on f is the composition map $f \circ h : M \to P$. For $f \in C^\infty(M,P)$ let Σ_f be the set of its critical points, and

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\},$$
$$\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}(M)\}$$

be respectively the *stabilizer* and the *orbit* of f under that action. Endow these spaces with C^∞ topologies and denote by $\mathcal{D}_{\mathrm{id}}(M)$ and $\mathcal{S}_{\mathrm{id}}(f)$ the corresponding path components of id_M in $\mathcal{D}(M)$ and $\mathcal{S}(f)$, and by $\mathcal{O}_f(f)$ the path component of $\mathcal{O}(f)$ containing f. We will omit X from notation whenever it is empty.

Notice that $\mathcal{S}_{\mathrm{id}}(f)$ is a normal subgroup of $\mathcal{S}(f)$, and the quotient:

$$\pi_0\mathcal{S}(f) := \mathcal{S}(f)/\mathcal{S}_{\mathrm{id}}(f)$$

is the group of path components of $\mathcal{S}(f)$. This group is an analogue of mapping class group for f-preserving diffeomorphisms.

Let $\mathcal{F}(M,P)$ be a subset of $C^\infty(M,P)$ consisting of maps satisfying the following two axioms:

- (B) The map f takes a constant value at each connected component of ∂M and has no critical points in ∂M;
- (L) For every critical point z of f, there are local coordinates in which f is a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ of degree ≥ 2 without multiple factors.

Evidently, $\mathcal{F}(M,P)$ contains all Morse maps.

For $f \in \mathcal{F}(M,P)$ the homotopy types of $\mathcal{S}_{\mathrm{id}}(f)$ and orbits were described by S. Maksymenko, and the homotopy types of connected components of orbit $\mathcal{O}(f)$ by S. Maksymenko, E. Kudryavtseva (for Morse maps and for smooth functions $f : M \to \mathbb{R}$ with *simple singularities* which are not homogeneous but quasi-homogeneous), B. Feshchenko, I. Kuznietsova, Yu. Soroka, A. Kravchenko.

Theorem 1. *Let $f \in \mathcal{F}(M,P)$. Then the natural map $p : \mathcal{S}(f) \to \pi_0\mathcal{S}(f)$ has a section:*

$$s : \pi_0\mathcal{S}(f) \to \mathcal{S}(f),$$

so s is a homomorphism such that $p \circ s = \mathrm{id}_{\pi_0\mathcal{S}_{\mathrm{id}}(f)}$.

Moreover, if M is orientable, then there exists a symplectic structure, i.e. a non-degenerate differential 2-form ω, on M, such that the image $s\big(\pi_0\mathcal{S}(f)\big)$ consists of symplectic diffeomorphisms with respect to ω.