Modified quaternionic operator calculus and its application to micropolar elasticity

Klaus Gürlebeck
(Chair of Applied Mathematics, Bauhaus-Universität Weimar, 99423 Weimar, Germany)
E-mail: klaus.guerlebeck@uni-weimar.de

Dmitrii Legatiuk
(Chair of Applied Mathematics, Bauhaus-Universität Weimar, 99423 Weimar, Germany)
E-mail: dmitrii.legatiuk@uni-weimar.de

Original ideas for the extension of classical elasticity theory to account microeffects of a continuum go back to the work [1] of Cosserat brothers, where they introduced a new theory called the *Cosserat continuum*. The introduced theory grabbed attention of many scientists. Among others, works of Eringen [2], and Nowacki [3] significantly supported further development of the theory. Eringen introduced micro-inertia in the theory, which has led to renaming of the theory to the *micropolar elasticity*. From practical point of view, the micropolar theory models not only displacements of a continuum, as in the classical theory of elasticity, but also its rotations.

In this talk we introduce representation formulae for the solution of spatial boundary value problems of micropolar elasticity. The representation formulae are constructed in the framework of quaternionic analysis, which is a natural extension of the classical complex analysis to higher dimensions. The main toolbox for constructing representation formulae for boundary value problems of mathematical physics in hypercomplex analysis is the co-called quaternionic operator calculus [4, 5]. The essential ingredient is the T-operator (Teodorescu transform), which is a right inverse to the generalised Cauchy-Riemann operator. Accomplishing the T-operator with the F-operator (Cauchy-Bitsadze operator), the higher-dimensional generalisation of the classical Borel-Pompeiu formula can be obtained.

In this talk, we consider the following boundary value problem:

Problem 1. Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. A boundary value problem of the micropolar elasticity is formulated as follows

$$(\lambda + 2\mu + \kappa)\,\nabla\nabla \cdot \mathbf{u} - (\mu + \kappa)\,\nabla \times \nabla \times \mathbf{u} = -\kappa \nabla \times \boldsymbol{\varphi}, \tag{1}$$

$$(\alpha + \beta + \gamma)\,\nabla\nabla \cdot \boldsymbol{\varphi} - \gamma \nabla \times \nabla \times \boldsymbol{\varphi} - 2\kappa\boldsymbol{\varphi} = -\kappa \nabla \times \mathbf{u}, \tag{2}$$

with boundary conditions

$$\left\{ \begin{array}{rcll} \mathbf{u} & = & \mathbf{g}_1 & \text{on } \Gamma_0, \\ \boldsymbol{\varphi} & = & \mathbf{g}_2 & \text{on } \Gamma_0, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcll} t_{lk} n_l & = & t_{(\mathbf{n})k} & \text{on } \Gamma_1, \\ m_{lk} n_l & = & m_{(\mathbf{n})k} & \text{on } \Gamma_1, \end{array} \right. \tag{3}$$

where $\mathbf{u}$ is the displacement vector, $\boldsymbol{\varphi}$ is the vector of micropolar rotation, t_{lk} is the stress tensor, m_{lk} is the couple stress tensor, ρ is the material density, j is a rotational inertia, λ and μ are the Lamé parameters, $\kappa, \alpha, \beta, \gamma$ are material parameters of micropolar theory, n_j are components of the unit outer normal vector, $t_{(\mathbf{n})k}$ are given surface forces, and $m_{(\mathbf{n})k}$ are given surface moments.

After that, a quaternionic formulation of the boundary value problem (1)-(3) is considered [6]:

Proposition 2. *Considering the displacement field $\mathbf{u} \in C^2(\Omega)$ and micropolar rotations $\boldsymbol{\varphi} \in C^2(\Omega)$ as pure quaternions, i.e. $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, $\boldsymbol{\varphi} = \varphi_1\mathbf{e}_1 + \varphi_2\mathbf{e}_2 + \varphi_3\mathbf{e}_3$, equations of micropolar elasticity (1)-(2) can be written as follows*

$$\begin{array}{rcl} D\,M_1\,D\,\mathbf{u} + \kappa \mathrm{Vec}\,D\boldsymbol{\varphi} & = & 0, \\ \left(D - i\sqrt{\frac{2\kappa}{\gamma}}\right) M_2 \left(D + i\sqrt{\frac{2\kappa}{\gamma}}\right) \boldsymbol{\varphi} + \kappa \mathrm{Vec}\,D\mathbf{u} & = & 0, \end{array} \tag{4}$$

where the operators M_1 and M_2 are defined by

$$\begin{array}{rcl} M_1\mathbf{w} & := & -(\lambda+2\mu+\kappa)w_0-(\mu+\kappa)w_1\mathbf{e}_1-(\mu+\kappa)w_2\mathbf{e}_2 \\ & & -(\mu+\kappa)w_3\mathbf{e}_3, \\ M_2\mathbf{w} & := & -(\alpha+\beta+\gamma)w_0-\gamma\, w_1\mathbf{e}_1-\gamma\, w_2\mathbf{e}_2-\gamma\, w_3\mathbf{e}_3, \end{array}$$

for a quaternion-valued function $\mathbf{w}=w_0+w_1\mathbf{e}_1+w_2\mathbf{e}_2+w_3\mathbf{e}_3$.

By reformulating the system as a system of operator equations, the questions of existence, regularity, stablity and uniqueness can be studied by using the classical and modified versions of quaternionic operator calculus [4, 5]:

Theorem 3. *The system of equations*

$$\left\{\begin{array}{rcl} D\,M_1\,D\,\mathbf{u}+\kappa\mathrm{Vec}\,D\boldsymbol{\varphi} & = & 0, \\ \left(D-i\sqrt{\frac{2\kappa}{\gamma}}\right)M_2\left(D+i\sqrt{\frac{2\kappa}{\gamma}}\right)\boldsymbol{\varphi}+\kappa\mathrm{Vec}\,D\mathbf{u} & = & 0, \end{array}\right. \tag{5}$$

with Dirichlet boundary conditions

$$\left\{\begin{array}{rcll} \mathbf{u} & = & \mathbf{g}_1 & \text{on } \Gamma_0, \\ \boldsymbol{\varphi} & = & \mathbf{g}_2 & \text{on } \Gamma_0, \end{array}\right.$$

is equivalent to the system of operator equations

$$\left\{\begin{array}{rcl} \mathbf{u} & = & F_\Gamma\tilde{\mathbf{g}}_1+TM_1^{-1}F_\Gamma(\mathrm{tr}\,TM_1^{-1}F_\Gamma)^{-1}Q_\Gamma\tilde{\mathbf{g}}_1 \\ & & -\kappa\,TM_1^{-1}T\,\mathrm{Vec}\,D\boldsymbol{\varphi}, \\ & & \\ \boldsymbol{\varphi} & = & F_\alpha\tilde{\mathbf{g}}_2+T_\alpha M_2^{-1}F_{-\alpha}\left(\mathrm{tr}\,T_\alpha M_2^{-1}F_{-\alpha}\right)^{-1}Q_\alpha\tilde{\mathbf{g}}_2 \\ & & -\kappa\,T_\alpha M_2^{-1}T_{-\alpha}\mathrm{Vec}\,D\mathbf{u}, \end{array}\right. \tag{6}$$

where $\tilde{\mathbf{g}}_1=\mathbf{g}_1+\kappa\,\mathrm{tr}\,TM_1^{-1}T\,\mathrm{Vec}\,D\boldsymbol{\varphi}$ *and* $\tilde{\mathbf{g}}_2=\mathbf{g}_1+\kappa\,\mathrm{tr}\,T_\alpha M_2^{-1}T_{-\alpha}\mathrm{Vec}\,D\mathbf{u}$.

Further results related to the uniqueness of solution, as well as the estimate for a difference between the micropolar model and the classical model of elasticity, will be presented in the talk.

References

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