Realization of groups as fundamental groups of orbits of smooth maps

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Let M be a connected compact oriented surface and P be a real line $\mathbb{R}$ or a circle S^1. Note, that the group $\mathcal{D}(M)$ of diffeomorphisms of M naturally acts on the space of smooth maps $C^\infty(M,P)$ by the rule $(f,h) \longmapsto f \circ h$, where $h \in \mathcal{D}(M)$, $f \in C^\infty(M,P)$. For $f \in C^\infty(M,P)$ denote by $\mathcal{O}(f)$ the orbit of f under this action. Let $\mathcal{M}(M,P)$ be the set of isomorphism clasess of fundamental groups $\pi_1\mathcal{O}(f)$ of orbits of Morse maps $f\colon M \to P$.

S. Maksymenko [1, 2] and B. Feshchenko [3] introduced the sets of isomorphism classes $\mathcal{B}$ and $\mathcal{T}$ of groups generated by direct products and certain wreath products. They have proved that $\mathcal{M}(M,P) \subset \mathcal{B}$ if M is different from a 2-sphere S^2 and a 2-torus T^2, and $\mathcal{M}(T^2,\mathbb{R}) \subset \mathcal{T}$. We proved that these inclusions are equalities.

Definition 1. Let $\mathcal{B}$ be a minimal class of groups satisfying the following conditions:

1) $1 \in \mathcal{B}$;
2) if $G_1, G_2 \in \mathcal{B}$, then $G_1 \times G_2 \in \mathcal{B}$;
3) if $G \in \mathcal{B}$ i $n \geq 1$, then $G \wr_n \mathbb{Z} \in \mathcal{B}$.

Let also $\mathcal{T}$ be the set of isomorphism classes of groups consisting of groups of the form $G \wr_{n,m} \mathbb{Z}^2$, where $G \in \mathcal{B}$ and $n, m \geq 1$.

Let also $\mathcal{B}^O$ be a subclass of $\mathcal{B}$ consisting of groups $(A \times B) \wr_n \mathbb{Z}$, where $A, B \in \mathcal{B} \setminus \{1\}$ and $n \geq 1$. Note, that $\mathcal{B}^O \subset \mathcal{B} \subset \mathcal{T}$.

Denote by $\mathcal{F}(M,P)$ the space of smooth maps $f \in C^\infty(M,P)$ satisfying the following two conditions:

(1) all critical points of f belong to the interior of M, and f takes constant values on each connected component of the boundary of M;

(2) for each critical point z of f its germ at z is smoothly equivalent to some non-zero homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ of degree ≥ 2 without multiple factors.

The set of all Morse maps from M to P is denoted by $Morse(M,P)$. For each map $f \in \mathcal{F}(M,P)$ we can define the (continuous) function ε_f from the set of connected components of the boundary ∂M to $\{\pm 1\}$, which takes the value -1 on the boundary component if f has a local minimum on this component, and $+1$ if f has a local maximum on this component. Let $\mathcal{E}_M$ be the set of all continuous functions $\varepsilon\colon \partial M \to \{\pm 1\}$. For $\varepsilon \in \mathcal{E}_M$ we denote by $\mathcal{F}(M,P,\varepsilon)$ ($Morse(M,P,\varepsilon)$) subset of $\mathcal{F}(M,P)$ ($Morse(M,P)$) of functions f, for which $\varepsilon_f = \varepsilon$.

Denote

$$\mathcal{G}_X(M,P,\varepsilon) := \{\pi_1\mathcal{O}(f,X) \mid f \in \mathcal{F}(M,P,\varepsilon)\},$$

$$\mathcal{M}_X(M,P,\varepsilon) := \{\pi_1\mathcal{O}(f,X) \mid f \in Morse(M,P,\varepsilon)\},$$

$$\mathcal{G}^\Psi := \{\pi_1\mathcal{O}(f) \mid f \in \mathcal{F}(T^2,\mathbb{R}), \text{the Kronrod-Reeb graph } \Gamma_f \text{ is a tree}\},$$

$$\mathcal{M}^\Psi := \{\pi_1\mathcal{O}(f) \mid f \in Morse(T^2,\mathbb{R}), \text{the Kronrod-Reeb graph } \Gamma_f \text{ is a tree}\},$$

$$\mathcal{G}^O := \{\pi_1\mathcal{O}(f) \mid f \in \mathcal{F}(T^2,\mathbb{R}), \text{the Kronrod-Reeb graph } \Gamma_f \text{ has an unique cycle}\},$$

$$\mathcal{M}^O := \{\pi_1\mathcal{O}(f) \mid f \in Morse(T^2,\mathbb{R}), \text{the Kronrod-Reeb graph } \Gamma_f \text{ has an unique cycle}\}.$$

Theorem 2. (1) *Let M be a connected compact oriented surface distinct from 2-torus and 2-sphere, and let $\varepsilon\colon \partial M \to \{\pm 1\}$ be an arbitrary map from $\mathcal{E}_M$. Then*

a) *if $M = S^1 \times [0,1]$, and ε is constant, i.e takes the same value on components of the boundary ∂M, then $\mathcal{M}_{\partial M}(M, P, \varepsilon) = \mathcal{G}_{\partial M}(M, P, \varepsilon) = \mathcal{B} \setminus \{1\}$,*

b) *if $M = S^1 \times [0,1]$ and ε takes different values on the components of the boundary ∂M or $M \neq S^1 \times [0,1]$, then $\mathcal{M}_{\partial M}(M, P, \varepsilon) = \mathcal{G}_{\partial M}(M, P, \varepsilon) = \mathcal{B}$.*

(2) *There are equalities $\mathcal{M}^{\Psi} = \mathcal{G}^{\Psi} = \mathcal{T}$, $\mathcal{M}^{O} = \mathcal{G}^{O} = \mathcal{B}^{O}$.*

References

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