Topology of spaces of smooth functions and gradient-like flows with prescribed singularities on surfaces

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Let M be a smooth orientable connected closed two-dimensional surface, and $f_0 \in \mathcal{F}(M)$ a function all whose critical points have types A_i, D_j, E_k. Consider the set $\mathcal{F} = \mathcal{F}(f_0)$ of all functions $f \in C^\infty(M)$ having the same types of local singularities as f_0. Denote by $\mathcal{D}^0(M)$ the component of unity in the group $\mathcal{D}(M) = \mathrm{Diff}^+(M)$ of orientation-preserving diffeomorphisms. The group $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$ acts on the space $\mathcal{F}$ by "left-right changes of coordinates".

We want to describe the topology of the space $\mathcal{F}$, equipped with the C^∞-topology, and its decomposition into $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$- and $\mathcal{D}^0(M)$-orbits. This problem was solved by the author in the cases when either f_0 is a Morse function and $\chi(M) < 0$ [2, 3], or all critical points of f_0 have A_μ types, $\mu \in \mathbb{N}$ [4]. Topology of the $\mathcal{D}^0(M)$-orbits was studied by S.I. Maksymenko [5] (allowing some other types of degenerate singularities) and by the author [2, 3, 4] (for A_μ-singularities).

For any function $f \in \mathcal{F}$, consider the set $\mathcal{C}_f := \{P \in M \mid \mathrm{d}f(P) = 0\}$ of its critical points. These critical points form five classes of topological equivalence (some classes may be empty):

$$\mathcal{C}_f^{min} = \bigcup_{i\ge1} A_{2i-1}^{+,+}(f),\ \mathcal{C}_f^{max} = \bigcup_{i\ge1} A_{2i-1}^{+,-}(f),\ \mathcal{C}_f^{saddle} = A_1^-(f) \cup \bigcup_{\eta=\pm}\big(\bigcup_{i\ge2} A_{2i-1}^{-,\eta}(f) \cup D_{2i+1}^{\eta}(f)\big) \cup E_7^{\eta}(f),$$

$$\mathcal{C}_f^{triv} = \big(\bigcup_{i\ge1,\eta=\pm} A_{2i}^{\eta}(f)\big) \cup \big(\bigcup_{j\ge2} D_{2j}^{+}(f)\big) \cup E_6^+(f) \cup E_6^-(f) \cup E_8^+(f) \cup E_8^-(f), \quad \mathcal{C}_f^{mult} = \bigcup_{j\ge2} D_{2j}^-(f),$$

i.e. the critial points of local minima, local maxima, saddle points, quasi- and multy-saddle points, respectively. Here $A_i^{\pm,\pm}(f)$, $D_j^{\pm}(f)$ and $E_k^{\pm}(f)$ denote the corresponding subsets of critical points of $A-D-E$ types. In the set $\mathcal{C}_f^{extr} := \mathcal{C}_f^{min} \cup \mathcal{C}_f^{max}$ of local extremum points, consider the subset $\mathcal{C}_f^{extr*}$ of degenerate (non-Morse) critical points.

Denote $s := \max\{0, \chi(M)+1\}$.

Theorem 1. *For any function $f_0 \in C^\infty(M)$, whose all critical points have $A-D-E$ types, the space $\mathcal{F} = \mathcal{F}(f_0)$ has the homotopy type of a manifold $\mathbb{B} = \mathbb{B}(f_0)$ having dimension $\dim \mathbb{B} = 2s + |\mathcal{C}_{f_0}^{extr}| + |\mathcal{C}_{f_0}^{extr*}| + 2|\mathcal{C}_{f_0}^{triv}| + 3|\mathcal{C}_{f_0}^{saddle}| + 4|\mathcal{C}_{f_0}^{mult}|$. Moreover:*

(a) *There exists a surjective submersion $\kappa : \mathcal{F} \to \mathbb{B}$ and a stratification (respectively, a fibration of codimension $|\mathcal{C}_{f_0}|$) on $\mathbb{B}$ such that every $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$-orbit (resp., $\mathcal{D}^0(M)$–orbit) in $\mathcal{F}$ is the κ-preimage of a stratum (resp., a fiber) in $\mathbb{B}$.*

(b) *The map κ provides a homotopy equivalence between any $\mathcal{D}^0(M)$-invariant subset $I \subseteq \mathcal{F}$ and its image $\kappa(I) \subseteq \mathbb{B}$. In particular, it provides homotopy equivalences between $\mathcal{F}$ and $\mathbb{B}$, and between every $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$–orbit (resp., $\mathcal{D}^0(M)$–orbit) from $\mathcal{F}$ and the corresponding stratum (resp., fiber) in $\mathbb{B}$.*

In particular, $\pi_k(\mathcal{F}) \cong \pi_k(\mathbb{B})$, $H_k(\mathcal{F}) \cong H_k(\mathbb{B})$. Thus $H_k(\mathcal{F}) = 0$ for all $k > \dim \mathbb{B}$.

Our proof of Theorem 1 uses a result (obtained in collaboration with Alexandra Orevkova) about a "uniform" reduction of a smooth function to a normal form near its critical points.

Suppose $\Omega \in \Lambda^n(M)$ is a volume form on a n-manifold $M = M^n$. Let $\mathcal{P} \subset M$ be a finite subset. For any vector field ξ on $M' := M \setminus \mathcal{P}$, we assign the $(n-1)$-form $\beta = i_\xi\Omega \in \Lambda^{n-1}(M')$. Clearly, this assignement is one-to-one. Furthermore, the flow of the vector field ξ is volume-preserving if and only

β is a closed form. Indeed: the Lie derivative $L_\xi\Omega = (i_\xi \mathrm{d} + \mathrm{d}i_\xi)\Omega = \mathrm{d}i_\xi\Omega = \mathrm{d}\beta$, so the Lie derivative vanishes if and only if $\mathrm{d}\beta = 0$. By abusing language, we will call the $(n-1)$-form β a *flow*.

Suppose now that $n = \dim M = 2$. A closed 1-form β on $M' = M \setminus \mathcal{P}$ will be called a *gradient-like flow* on M if there exists a Morse function $f \in C^\infty(M)$, called an *energy function* of β, such that

(i) the set $\mathcal{P}$ coincides with the set of local extremum points of f,
(ii) the 3-form $\mathrm{d}f \wedge \beta|_{M\setminus\mathcal{C}_f}$ has no zeros and defines a positive orientation on M,
(iii) in a neighbourhood of every point $P \in \mathcal{C}_f$, there exist local coordinates x, y such that either $\beta = \mathrm{d}(xy)$, $f = f(P) + x^2 - y^2$ and $P \in \mathcal{Z} = \mathcal{C}_f \setminus \mathcal{P}$, or $\beta = (x\mathrm{d}y - y\mathrm{d}x)/(x^2+y^2)$, $f = f(P) \pm (x^2+y^2)$ and $P \in \mathcal{P}$.

Geometrically, the set $\mathcal{P}_\beta := \mathcal{P}$ consists of *sourses* and *sinks* of the flow β and coincides with the set of local extremum points of the energy function f, while the set $\mathcal{Z}_\beta := \mathcal{Z} = \mathcal{C}_f \setminus \mathcal{P}$ consists of *saddle points* of the flow β and coincides with the set of saddle critical points of f.

Denote by $\mathcal{B}(M)$ the space of all gradient-like flows β on M (having arbitrary finite sets $\mathcal{Z} = \mathcal{Z}_\beta$ and $\mathcal{P} = \mathcal{P}_\beta$ of saddles, sourses and sinks depending on β). Endow this space with C^∞-topology. For a gradient-like flow $\beta_0 \in \mathcal{B}(M)$, denote by $\mathcal{B}(\beta_0)$ the set of all gradient-like flows $\beta \in \mathcal{B}(M)$ having the same local singularities as β_0 (in particular, $|\mathcal{Z}_\beta| = |\mathcal{Z}_{\beta_0}|$ and $|\mathcal{P}_\beta| = |\mathcal{P}_{\beta_0}|$).

We want to describe the topology of the space $\mathcal{B}(\beta_0)$, equipped with the C^∞-topology, and its decomposition into $\mathcal{D}^0(M)$-orbits and into classes of (orbital) topological equivalence.

Theorem 2. *For any gradient-like flow β_0 on M, the space $\mathcal{B}(\beta_0)$ has the homotopy type of the manifold $\mathbb{B} = \mathbb{B}(f_0)$ from Theorem 3, where f_0 is an energy function of β_0. Moreover:*

(a) *There exists a surjective submersion $\lambda : \mathcal{B}(\beta_0) \to \mathbb{B}$, a stratification and a $(|\mathcal{P}_{\beta_0}| + |\mathcal{Z}_{\beta_0}|)$-dimensional fibration on $\mathbb{B}$ such that every class of topological equivalence (resp., every $\mathcal{D}^0(M)$-orbit) in $\mathcal{B}(\beta_0)$ is the λ-preimage of the stratum (resp., the fibre) from $\mathbb{B}$.*
(b) *The map λ provides a homotopy equivalence between every $\mathcal{D}^0(M)$-invariant subset $I \subseteq \mathcal{B}(\beta_0)$ and its image $\lambda(I) \subseteq \mathbb{B}$. In particular, it provides a homotopy equivalence between $\mathcal{B}(\beta_0)$ and $\mathbb{B}$, as well as between every class of topological equivalence (resp., every $\mathcal{D}^0(M)$–orbit) in $\mathcal{B}(\beta_0)$ and the corresponding stratum (resp., fibre) in $\mathbb{B}$. All fibres and strata in $\mathbb{B}$ (and, thus, all classes of topological equivalence and all $\mathcal{D}^0(M)$–orbits in $\mathcal{B}(\beta_0)$) are homotopy equivalent either to a point, or to T^2, or to $SO(3)/G$ or to S^2, in dependence on whether $\chi(M) < 0$, or $\chi(M) = 0$, or $\chi(M) \cdot |\mathcal{Z}_{\beta_0}| > 0$, or $|\mathcal{Z}_{\beta_0}| = 0$, respectively, where G is a finite subgroup of $SO(3)$.*

In particuar, $\pi_k(\mathcal{B}(\beta_0)) \cong \pi_k(\mathbb{B})$, $H_k(\mathcal{B}(\beta_0)) \cong H_k(\mathbb{B})$. Thus $H_k(\mathcal{B}(\beta_0)) = 0$ for all $k > \dim \mathbb{B}$.

We will illustrate our results on several examples.

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References

[1] Elena A Kudryavtseva, Dmitry A Permyakov. Framed Morse functions on surfaces. *Sbornik Math.*, 201(4) : 501–567, 2010.
[2] Elena A Kudryavtseva. The Topology of Spaces of Morse Functions on Surfaces. *Math. Notes*, 92(2) : 219–236, 2012. http://arxiv.org/abs/1104.4792.
[3] Elena A Kudryavtseva. Special framed Morse functions on surfaces. *Moscow Univ. Math. Bull.*, 67(4) : 14–20, 2012. arXiv:1106.3116.
[4] Elena A Kudryavtseva. Topology of the spaces of functions with prescribed singularities on surfaces. *Doklady Mathematics*, 93(3) : 264–266, 2016.
[5] Sergiy I Maksymenko. Homotopy types of stabilizers and orbits of Morse functions on surfaces. *Ann. Glob. Anal. Geom.* 29(3) : 241–285, 2006. arXiv:math.GT/0310067.