Elements of probability theory and measures with values in hypercomplex algebras

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In recent years, the expansion of probability theory and measure theory from real values to values in hypercomplex numbers are actively studied because of their possible applications in mathematics and physics [1] – [5]. In this paper, we extend the notion of probability measure to the case where the measure takes values in the algebra of bihyperbolic numbers [6]. In addition, the concept of the real-valued measure is generalized to the quaternionic-valued measure [7].

The bihyperbolic numbers forms a 4-dimensional algebra over the field of real numbers $\mathbb{W}_4 = \{a_0 + a_1 e + a_2 f + a_3 g,\ a_i \in \mathbb{R},\ i = 0,1,2,3\}$ with basis $\{1, e, f, g\}$ and the following multiplications $e^2 = f^2 = g^2 = 1,\ ef = fe = g,\ eg = ge = f,\ fg = gf = e$.

Lemma 1. [8] *Any bihyperbolic number α can be represented as $\alpha = r_1 i_1 + r_2 i_2 + r_3 i_3 + r_4 i_4$, where i_k are idempotents of algebra $\mathbb{W}_4$, $r_k \in \mathbb{R}$, $k = 1,2,3,4$.*

We define on $\mathbb{W}_4$ the relation of partial order $\preccurlyeq$ such as $\alpha \preccurlyeq \beta \Longleftrightarrow \beta - \alpha \in \mathbb{W}_4^+ = \{x_1 i_1 + x_2 i_2 + x_3 i_3 + x_4 i_4 |\ x_k \geq 0,\ k = 1,2,3,4\}$. If $\alpha \preccurlyeq \beta$ but $\alpha \neq \beta$, we denote $\alpha \prec \beta$. Let us denote by A_x, the set of all bihyperbolic numbers which are not $\mathbb{W}_4$-comparable with $x \in \mathbb{W}_4$.

Definition 2. The $\mathbb{W}_4$-valued modulus of a bihyperbolic number $\alpha = r_1 i_1 + r_2 i_2 + r_3 i_3 + r_4 i_4$ is said to be $|\alpha|_{\mathbb{W}_4} = |r_1 i_1 + r_2 i_2 + r_3 i_3 + r_4 i_4|_{\mathbb{W}_4} = |r_1|\, i_1 + |r_2|\, i_2 + |r_3|\, i_3 + |r_4|\, i_4 \in \mathbb{W}_4^+$, where $|r_1|$, $|r_2|$, $|r_3|$, $|r_4|$ are ordinary modules of real numbers.

Definition 3. Let (Ω, Σ) be a measurable space. The function $P_{\mathbb{W}_4} : \Sigma \to \mathbb{W}_4$ is called a $\mathbb{W}_4$-valued probability (or bihyperbolic probability) on the σ-algebra of events Σ, if the following conditions hold: 1) $P_{\mathbb{W}_4}(A) \succcurlyeq 0, \forall A \in \Sigma$; 2) $P_{\mathbb{W}_4}(\Omega) = \zeta$, where $\zeta = 1, i_1, i_2, i_3, i_4$; 3) For any sequence $\{A_n,\ n \geq 1\} \subset \Sigma$ of pairwise incompatible random events we have $P_{\mathbb{W}_4}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P_{\mathbb{W}_4}(A_n)$.

The triplet $(\Omega, \Sigma, P_{\mathbb{W}_4})$ is called a $\mathbb{W}_4$-probability space.

Each $\mathbb{W}_4$-valued probability measure can be written in the form $P_{\mathbb{W}_4}(A) = P_1(A)\, i_1 + P_2(A)\, i_2 + P_3(A)\, i_3 + P_4(A)\, i_4$, where $P_1(A), P_2(A), P_3(A), P_4(A)$ are probabilities.

The topology induced by the bihyperbolic norm generates the Borel σ-algebra $\mathfrak{B}_{\mathbb{W}_4}$ in $\mathbb{W}_4$.

Definition 4. Let $(\Omega, \Sigma, P_{\mathbb{W}_4})$ be a $\mathbb{W}_4$-probability space. A function $X_{\mathbb{W}_4}(\omega) : \Omega \to \mathbb{W}_4$ such as $X_{\mathbb{W}_4}^{-1}(A) \in \Sigma$ for each open set A in $\mathbb{W}_4$ is called a $\mathbb{W}_4$-valued random variable.

Each $\mathbb{W}_4$-valued random variable $X_{\mathbb{W}_4}(\omega)$ can be written in the following form $X_{\mathbb{W}_4}(\omega) = X_1(\omega)\, i_1 + X_2(\omega)\, i_2 + X_3(\omega)\, i_3 + X_4(\omega)\, i_4$, where $X_1(\omega), X_2(\omega), X_3(\omega), X_4(\omega)$ are $\mathbb{R}$-random variables on Ω.

Theorem 5. *The $\mathbb{W}_4$-valued function $X_{\mathbb{W}_4}(\omega)$ on a measurable space (Ω, Σ) is a $\mathbb{W}_4$-valued random variable if and only if $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \prec x$ or $X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$ for all $x \in \mathbb{W}_4$.*

Theorem 6. *Let $X_{\mathbb{W}_4}(\omega)$ be a $\mathbb{W}_4$-valued random variable on $(\Omega, \Sigma, P_{\mathbb{W}_4})$. For $\forall x \in \mathbb{W}_4$ the following conditions are equivalent: $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \preccurlyeq x\} \in \Sigma$; $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \succ x$ or $X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$; $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \succcurlyeq x$ or $X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$; $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \prec x\} \in \Sigma$.*

The algebra of quaternions is a structure of the form $\mathbb{H}=\{a_0+a_1i+a_2j+a_3k,\ a_i\in\mathbb{R},\ i=0,1,2,3\}$, where $i^2=j^2=k^2=-1,\ ij=-ji=k,\ jk=-kj=i,\ ki=-ik=j.$

Definition 7. Let $\mathfrak{M}$ be a σ-algebra of subsets of a non-empty set X. A quaternionic measure ω on a measurable space $(X,\mathfrak{M})$ is a quaternion-valued function on $\mathfrak{M}$ such that for any collection of sets $\{A_n, n\in\mathbb{N}\}\subset\mathfrak{M}$ that $A_n\cap A_m=\emptyset$ whenever $n\neq m$ we have $\omega\left(\bigcup_{n=1}^{\infty}A_n\right)=\sum_{n=1}^{\infty}\omega\left(A_n\right)$.

Definition 8. The function of the sets $\operatorname{var}[\omega](A):=\sup\sum_{n=1}^{\infty}|\omega(A_n)|$ is defined on the $\mathfrak{M}$, where the supremum is taken for all partitions of A, we call the complete variation ω.

It is clear that $|\omega(A)|\leq \operatorname{v}ar[\omega](A)$.

Theorem 9. *The total variation* $\operatorname{var}[\omega]$ *of a quaternionic measure* ω *on a measurable space* $(X,\mathfrak{M})$ *is a positive measure on* $(X,\mathfrak{M})$.

Theorem 10. *If* ω *is a quaternionic measure on a measurable space* $(X,\mathfrak{M})$, *then* $\operatorname{var}[\omega](X)<\infty$.

Definition 11. Let μ be a positive measure and ω be a quaternionic measure on a measurable space $(X,\mathfrak{M})$. We say that ω is absolutely continuous with respect to μ if $\mu(A)=0$ implies $\omega(A)=0$ for $A\in\mathfrak{M}$. We write $\omega\ll\mu$.

Definition 12. Given a quaternionic measure ω on a measurable space $(X,\mathfrak{M})$, assume that there is a set $F\in\mathfrak{M}$ such that $\omega(A)=\omega(A\cap F)$ for every $A\in\mathfrak{M}$, we say that ω is concentrated on F. This is equivalent to say that $\omega(A)=0$ whenever $A\cap F=0$.

Let ω_1,ω_2 be quaternionic measures on $\mathfrak{M}$ and suppose there exist a pair of disjoint sets F,G such that ω_1 is concentrated on F and ω_2 is concentrated on G. Then we say that ω_1 and ω_2 are mutually singular, and write $\omega_1\perp\omega_2$.

Theorem 13. *Let* λ *be a signed real* σ*-finite measure on a measurable space* $(X,\mathfrak{M})$ *and let* ω *be a quaternionic measure on* $(X,\mathfrak{M})$. *Then there exists a unique pair of quaternionic measures* ω_a *and* ω_s *such that* $\omega=\omega_a+\omega_s,\ \omega_a\ll\lambda,\ \omega_s\perp\lambda$. *The pair* ω_a,ω_s *is called the Lebesgue decomposition of* ω *w.r.t.* λ, *where* ω_a *is the absolutely continuous part and* ω_s *is the singular part of the decomposition.*

Theorem 14. *(Radon-Nikodym). Let* μ *be a positive* σ*-finite measure on a measurable space* $(X,\mathfrak{M})$, *let* ω *be a quaternionic measure on* $(X,\mathfrak{M})$ *and let* ω_a *be absolutely continuous part of the Lebesgue decomposition of* ω *w.r.t.* μ. *Then there is a measurable quaternionic function* $h(x)$ *on* X *such that for every set* $A\in\mathfrak{M}$ $\omega_a(A)=\int_A h(x)\,d\mu$, *where* $h(x)$ *is uniquely defined up to a* μ*-null set.*

References

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