Perfect metrizability of the functor of idempotent measures

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Let $\mathbb{R}$ be the real line. The set $\mathbb{R}\cup\{-\infty\}$ considered with operations: addition $\oplus$ and multiplication $\odot$ defined as $u\oplus v=\max\{u,\,v\}$ and $u\odot v=u+v$, denotes by $\mathbb{R}_{\max}$. Let X be a compact Hausdorff space, $C(X)$ the algebra of continuous functions $\varphi\colon X\to\mathbb{R}$ with the usual algebraic operations. On $C(X)$ the operations $\oplus$ and $\odot$ we define as $\varphi\oplus\psi=\max\{\varphi,\,\psi\}$, $\varphi\odot\psi=\varphi+\psi$, $\lambda\odot\varphi=\varphi+\lambda_X$ here $\varphi,\,\psi\in C(X)$, $\lambda\in\mathbb{R}$. Recall [1] that a functional $\mu\colon\, C(X)\to\mathbb{R}$ is said to be an *idempotent probability measure* on X, if: 1) $\mu(\lambda_X)=\lambda$ for each $\lambda\in\mathbb{R}$; 2) $\mu(\lambda\odot\varphi)=\mu(\varphi)+\lambda$ for all $\lambda\in\mathbb{R}$, $\varphi\in C(X)$; 3) $\mu(\varphi\oplus\psi)=\mu(\varphi)\oplus\mu(\psi)$ for every $\varphi,\,\psi\in C(X)$. The set of all idempotent probability measures on X we denote by $I(X)$. Consider $I(X)$ as a subspace of $\mathbb{R}^{C(X)}$. The topological space $I(X)$ is compact [1]. For a given map $f\colon\, X\to Y$ of compact Hausdorff spaces the map $I(f)\colon\, I(X)\to I(Y)$ defines by the formula $I(f)(\mu)(\varphi)=\mu(\varphi\circ f)$, $\mu\in I(X)$, where $\varphi\in C(Y)$. The construction I is a normal covariant functor, acting in the category of compact Hausdorff spaces and their continuous maps. For $\mu\in I(X)$ we may define the support of μ : $\operatorname{supp}\mu=\cap\{A\subset X:\,\bar{A}=A,\,\mu\in I(A)\}$. For a point $x\in X$ by the rule $\delta_x(\varphi)=\varphi(x)$, $\varphi\in C(X)$, we define the Dirac measure δ_x supported on the singleton $\{x\}$.

Put

$$U_S(X)=\big\{\lambda\colon\, X\to[-\infty,\,0]\,\big|\quad \lambda \text{ is upper semicontinuous and there exists a } x_0\in X \text{ such that } \lambda(x_0)=0\big\}.$$

Then we have

$$I(X)=\left\{\bigoplus_{x\in X}\lambda(x)\odot\delta_x\,:\,\lambda\in U_S(X)\right\}.$$

We define a subset

$$I_\omega(X)=\left\{\bigoplus_{x\in X}\lambda(x)\odot\delta_x\,:\,\lambda\in U_S(X),\,|\{x\in X\,:\,\lambda(x)>-\infty\}|<\infty\right\}\subset I(X).$$

$I_\omega(X)$ is everywhere dense in $I(X)$ [**?**, **?**]. Put

$$\rho_2(\mu_1,\,\mu_2)=\inf\left\{\frac{\sum\limits_{(x,\,y)\in\operatorname{supp}\xi}\mathrm{e}^{\lambda_1(x)+\lambda_2(y)}\cdot\rho(x,\,y)}{\sum\limits_{x\in\operatorname{supp}\mu_1}\mathrm{e}^{\lambda_1(x)}\cdot\sum\limits_{y\in\operatorname{supp}\mu_2}\mathrm{e}^{\lambda_2(y)}}\,:\,\xi\in\Lambda_{1\,2}\right\},$$

where $\mu_i=\bigoplus\limits_{x\in X}\lambda_i(x)\odot\delta_x\in I_\omega(X)$, $i=1,\,2$. Further, for every pair $\mu,\,\nu\in I(X)$ take consequences $\{\mu_n\}$, $\{\nu_n\}\subset I_\omega(X)$ such that $\lim\limits_{n\to\infty}\mu_n=\mu$ and $\lim\limits_{n\to\infty}\nu_n=\nu$, and put

$$\rho_I(\mu,\,\nu)=\lim_{n\to\infty}\rho_2(\mu_n,\nu_n).$$

The function ρ_I is a metric on $I(X)$ generating the pointwise convergence topology on $I(X)$ and the restriction of which coincides with the metric ρ on X.

Consider a system ψ consisting of all mapgs $\psi_X\colon\, I^2(X)\to I(X)$, acting as the following. Given $M\in I^2(X)$ put $\psi_X(M)(\varphi)=M(\overline{\varphi})$, where for any function $\varphi\in C(X)$ the function $\overline{\varphi}\colon\, I(X)\to\mathbb{R}$ defines by the formula $\overline{\varphi}(\mu)=\mu(\varphi)$. Fix a compactum X and for a positive integer n put $\psi_{n+1,\,n}=\psi_{I^{n-1}(X)}\colon\, I^{n+1}(X)\to I^n(X)$. Note that $\psi_{n+1,\,n}\circ\eta_{n,\,n+1}=Id_{I^n(X)}$.

Lemma 1. $\psi_{1,0}\colon (I^2(X),\, \rho_{I^2}) \to (I(X),\, \rho_I)$ is a non-expanding map.

Lemma 2. For each $N \in \psi_{1,0}^{-1}(\mu)$ we have $\rho_I(\mu,\, \delta_{x_0}) = \rho_{I^2}(\delta_{\delta_{x_0}},\, N)$.

Lemma 3. If $\rho_I(\mu,\, \eta_{0,1}(X)) \geq \varepsilon$ then $\rho_{I^2}(I(\eta_{0,1})(\mu),\; \eta_{1,2}(I(X))) \geq \varepsilon$.

Theorem 4. The functor I is perfect metrizable.

References

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