On controllability problems for the heat equation with variable coefficients controlled by the Dirichlet boundary condition on a half-axis

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Consider the heat equation

$$w_t = \frac{1}{\rho}\left(k w_x\right)_x + \gamma w, \qquad x \in (0,+\infty),\ t \in (0,T), \tag{1}$$

controlled by the Dirichlet boundary condition

$$w(0,\cdot) = u, \qquad t \in (0,T), \tag{2}$$

under the initial condition

$$w(\cdot,0) = w^0, \qquad x \in (0,+\infty), \tag{3}$$

where $T = \text{const} > 0$; ρ, k, γ, w^0 are given functions; $u \in L^\infty(0,T)$ is a control. We assume $\rho, k \in C^1[0,+\infty)$, $\rho > 0$ and $k > 0$ on $[0,+\infty)$, $(\rho k) \in C^2[0,+\infty)$, $(\rho k)'(0) = 0$, and

$$\sigma(x) = \int_0^x \sqrt{\rho(\mu)/k(\mu)}\, d\mu \to +\infty \qquad \text{as } x \to +\infty.$$

In addition, we assume

$$(P(k,\rho) - \gamma) \in L^\infty(0,+\infty) \bigcap C^1[0,+\infty) \qquad \text{and} \qquad \sigma\sqrt{\frac{\rho}{k}}\,(P(k,\rho)-\gamma) \in L^1(0,+\infty),$$

where $P(k,\rho) = \frac{1}{4}\sqrt{\frac{k}{\rho}}\left(\sqrt{\frac{k}{\rho}}\left(\frac{k'}{k}+\frac{\rho'}{\rho}\right)\right)' + \left(\frac{1}{4}\sqrt{\frac{k}{\rho}}\left(\frac{k'}{k}+\frac{\rho'}{\rho}\right)\right)^2$.

Control system (1)–(3) is considered in modified Sobolev spaces. Denote $\eta = (k\rho)^{1/4}$, $\theta = \left(\frac{k}{\rho}\right)^{1/4}$, $\mathcal{D}_{\eta\theta} = \theta^2\left(\frac{d}{dx} + \frac{\eta'}{\eta}\right)$. Denote also $\mathbb{R}_+ = (0,+\infty)$. For $p = 1,2$, denote

$$\widetilde{\mathbb{H}}_+^0 = \left\{\varphi \in L^2_{\text{loc}}(\mathbb{R}_+) \mid \left(\frac{\eta}{\theta}\varphi\right) \in L^2(\mathbb{R}_+)\right\}, \quad \widetilde{\mathbb{H}}_+^p = \left\{\varphi \in \widetilde{\mathbb{H}}_+^{p-1} \mid \left(\frac{\eta}{\theta}\mathcal{D}_{\eta\theta}^p\varphi\right) \in L^2(\mathbb{R}_+) \text{ and } \varphi(0) = 0\right\},$$

with the norm

$$[\![\varphi]\!]_+^s = \sqrt{\sum_{m=0}^s \left(\left\|\frac{\eta}{\theta}\left(\mathcal{D}_{\eta\theta}^m\varphi\right)\right\|_{L^2(\mathbb{R}_+)}\right)^2}, \qquad \varphi \in \widetilde{\mathbb{H}}_+^s, \quad s = 0,1,2,$$

and the dual spaces $\widetilde{\mathbb{H}}_+^{-s} = \left(\widetilde{\mathbb{H}}_+^s\right)^*$ with the norms associated with the strong topology of these spaces, $s = 0,1,2$.

We suppose $\left(\frac{d}{dt}\right)^p w : [0,T] \to \widetilde{\mathbb{H}}_+^{-2p}$, $p = 0,1$, and $w^0 \in \widetilde{\mathbb{H}}_+^0$ in system (1)–(3).

Let $T>0$ and $w^0 \in \widetilde{\mathbb{H}}_+^0$. By $\mathcal{R}_T(w^0)$, denote the set of all states $w^T \in \widetilde{\mathbb{H}}_+^0$ for which there exists a control $u \in L^\infty(0,T)$ such that for the solution w to system (1)–(3) we have $w(\cdot,T)=w^T$.

Definition 1. A state $w^0 \in \widetilde{\mathbb{H}}_+^0$ is said to be controllable to a state $w^T \in \widetilde{\mathbb{H}}_+^0$ with respect to system (1)–(3) in a given time $T>0$ if $w^T \in \mathcal{R}_T(w^0)$.

Definition 2. A state $w^0 \in \widetilde{\mathbb{H}}_+^0$ is said to be null-controllable with respect to system (1)–(3) in a given time $T>0$ if $0 \in \mathcal{R}_T(w^0)$.

Definition 3. A state $w^0 \in \widetilde{\mathbb{H}}_+^0$ is said to be approximately controllable to a state $w^T \in \widetilde{\mathbb{H}}_+^0$ with respect to system (1)–(3) in a given time $T>0$ if $w^T \in \overline{\mathcal{R}_T(w^0)}$, where the closure is considered in the space $\widetilde{\mathbb{H}}_+^0$.

Consider also control system with the simplest heat operator (the case $\rho=k=1$, $\gamma=0$)

$$z_t = z_{yy}, \qquad y \in (0,+\infty),\ t\in(0,T), \tag{4}$$

$$z(0,\cdot)=v, \qquad t\in(0,T), \tag{5}$$

$$z(\cdot,0)=z^0, \qquad y\in(0,+\infty), \tag{6}$$

where $v \in L^\infty(0,T)$ is a control, $\left(\frac{d}{dt}\right)^m z : [0,T] \to \widetilde{H}_+^{-2m}$, $m=0,1$, $w^0 \in \widetilde{H}_+^0 = L^2(\mathbb{R}_+)$. Here

$$\widetilde{H}_+^s = \left\{\varphi \in L^2(\mathbb{R}_+) \mid \left(\forall k=\overline{0,s}\ \varphi^{(k)} \in L^2(\mathbb{R}_+)\right) \text{ and } \left(\forall k=\overline{0,s-1}\ \varphi^{(k)}(0^+)=0\right)\right\}, \quad s=0,1,2,$$

with the norm

$$\|\varphi\|_+^s = \sqrt{\sum_{k=0}^{s} \binom{s}{k} \left(\left\|\varphi^{(k)}\right\|_{L^2(\mathbb{R}_+)}\right)^2}, \qquad \varphi \in \widetilde{H}_+^s, \quad s=0,1,2,$$

and $\widetilde{H}_+^{-s} = \left(\widetilde{H}_+^s\right)^*$ with the norms associated with the strong topology of these spaces, $s=0,1,2$.

Controllability problems for system (4)–(6) were investigated in [1].

To study controllability problems for system (1)–(3), we use the transformation operator $\widetilde{\mathbb{T}} : \widetilde{H}_+^{-2} \to \widetilde{\mathbb{H}}_+^{-2}$. It was introduced and studied in [2]. In particular, it has been proved therein that $\widetilde{\mathbb{T}}$ is a continuous one-to-one mapping between the spaces $\widetilde{H}^s$ and $\widetilde{\mathbb{H}}^s$, $s=-2,-1,0$.

In the present talk, we prove that the transformation operator $\widetilde{\mathbb{T}}$ is one-to-one mapping between the sets of the solutions to system (4)–(6) and to system (1)–(3). The application of the operator $\widetilde{\mathbb{T}}$ allows us to conclude that the control system (1)–(3) replicates the controllability properties of the control system (4)–(6) and vice versa. A relation between controls u and v is also found. Thus, we obtain the following main results.

Theorem 4. *If a state $w^0 \in \widetilde{\mathbb{H}}_+^0$ is null-controllable with respect to system (1)–(3) in a time $T>0$, then $w^0=0$.*

Theorem 5. *Each state $w^0 \in \widetilde{\mathbb{H}}_+^0$ is approximately controllable to any target state $w^T \in \widetilde{\mathbb{H}}_+^0$ with respect to system (1)–(3) in a given time $T>0$.*

References

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[2] Larissa V. Fardigola, Transformation Operators in Controllability Problems for the Wave Equations with Variable Coefficients on a Half-Axis Controlled by the Dirichlet Boundary Condition. *Math. Control. Relat. Fields*, 5(1) : 31–53, 2015.