Integration over non-rectifiable curves: spirals of high torsion

David B. Katz
(Moscow, Udaltsova, 16, 119415)
E-mail: katzdavid89@gmail.com

The presentation is devoted to some new results related to the classical problem of complex analysis - the Riemann boundary value problem. We, however, consider it on a non-rectifiable curve and pay considerable attention to the geometric features of the curve. In this case, we will talk about spirals, which we will classify depending on the speed of it's twisting. We consider the geometric properties of arcs in the neighbourhood of their ends.

Let us consider first a well known boundary-value problem of complex analysis – so called Riemann problem on a simple Jordan arc (see, for instance, [1, 2, 3]).

Given a directed Jordan arc Γ in the complex plane $\mathbb{C}$ with beginning and end at points a_1 and a_2 relatively, and two functions $G(t)$, $g(t)$, $t \in \Gamma$. Find all holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ functions $\Phi(z)$ which vanish at ∞ and have boundary limits $\Phi^{\pm}(t)$ from the left and from the right correspondingly at any point $t \in \Gamma \setminus \{a_1, a_2\}$ such that

$$\Phi^{+}(t) = G(t)\Phi^{-}(t) + g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}. \tag{1}$$

In addition, the desired function Φ must satisfy certain conditions on its growth at the end points $a_{1,2}$.

In numerous classical works (see [1, 2, 3] and many other) the solutions of this problem are obtained in terms of Cauchy type integrals. Particularly, a solution of so called jump problem

$$\Phi^{+}(t) = \Phi^{-}(t) + g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}, \tag{2}$$

on piecewise - smooth arc Γ is representable as the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g(t)\,dt}{t - z}, \quad z \notin \Gamma. \tag{3}$$

As a result, in all classical works on this problem the boundary Γ is assumed rectifiable, although the formulation of the problem does not imply this restriction. It keeps the sense for non-rectifiable arcs, too. The Riemann boundary-value problem for non-rectifiable boundaries was solved first in the papers [4, 5, 6, 7].

We introduce the concept of torsion of arc Γ [8]. The torsion of arc Γ at point a_j, $j = 1, 2$ is a value

$$\tau_j := \inf \left\{ p > 0 : \iint |K_{\Gamma}(z)|^{1/p} dx\, dy < \infty \right\},$$

where integral is taken over a neighborhood of a_j. If $\tau_j < 1$, then we say that the arc has moderate torsion at point a_j, otherwise its torsion is high. This value characterizes the rate of curling of Γ around the point a_j.

As it appears, the torsion concept is very closely connected with the integrator concept that we introduced earlier - and this allows us to get some new results in this geometric terms.

References

[1] Gakhov F.D. *Boundary value problems.* Oxford : Pergamon Press, 1966.
[2] Muskhelishvili N.I. *Singular Integral Equations.* Dordrecht : Springer, 1958.
[3] Jian-Ke Lu. *Boundary Value Problems for Analytic Functions.* Singapore : World Scientific Publishing Company, 1994.
[4] Boris Kats. Riemann boundary-value problem on non-rectifiable Jordan curve. *Doklady AN USSR*, 267(4) : 789–792, 1982.
[5] Boris Kats. Riemann problem on closed Jordan curve. *Izvestiya vuzov. Matem.*, 4 : 68–80, 1983.

[6] Boris Kats. Riemann problem on non-closed Jordan curve. *Izvestiya vuzov. Matem.*, 12 : 30–38, 1983.
[7] Boris Kats. On inhomogeneous Riemann problem on non-closed Jordan curve. *Trudy semin. po kraevym zadacham*, 21 : 87–93, 1984.
[8] David B. Katz, Boris Kats Torsions and integrations. *Complex Variables and Elliptic Equations* , 2021.