On some properties of moduli of smoothness of conformal mapping of simply connected domains

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Let consider a function realizing homeomorphism of the closed unit disk onto the closure of simply connected domain in the complex plane bounded by a smooth Jordan curve conformal in the open unit disk. Let suppose that boundary of this domain is characterized by the angle between the tangent to the curve and the positive real axis considered as the function of the arc length on the curve.

O. D. Kellogg in 1912 proved the theorem in which it had been established that if this angle function satisfies Holder condition, then the derivative of the function realizing mapping of unit disk onto the closure of the given domain satisfies Holder condition with the same index. Connection between properties of the boundary of the domain and properties of the considered function was investigated in works by numerous authors: S. E. Warshawski, J. L. Geronimus, S. J. Alper, R. N. Kovalchuk, L. I. Kolesnik, P. M. Tamrazov (more detailed see [1–3], [5] and [6]). Some close problems were investigated by V. A. Danilov, E. P. Dolzenko, E. M. Dynkin, N. A. Shirokov, S. R. Bell and S. G. Krantz, V. V. Andrievskii, V. I. Belyi, B. Oktay, D. M. Israfilov and others (more detailed see [3–5] and [7]).

Certain results in terms of moduli of smoothness of different types (uniform curvilinear, arithmetic, local and integral moduli of smoothness of arbitrary order) were received by author. In particular, some estimates for integral moduli of smoothness were considered in [4–6].

Let $\omega_{k,z}(f(z),\delta)$ be a noncentralized local arithmetic modulus of smoothness of order k ($k \in \mathbb{N}$) of the function $w = f(z)$ at a point z on the curve γ [1]. Let consider the integral modulus of smoothness of order k for the function $w = f(z)$ on the curve γ introduced in [2] by the formula $\widehat{\omega}_k(f(z),\delta) = \left(\int\limits_\gamma [\omega_{k,z}(f(z),\delta)]^p \, d\lambda(z)\right\}^{1/p}$, $1 \le p < +\infty$, $k \in \mathbb{N}$, where $\lambda = \lambda(z)$ is the linear Lebesgue's measure on the curve.

Let G_1 and G_2 be the simply connected domains in the complex plane bounded by the smooth Jordan curves Γ_1 and Γ_2. Let $\tau_1(s_1)$ be the angle between the tangent to Γ_1 and the positive real axis, $s_1(z)$ be the arc length on Γ_1. Let $\tau_2(s_2)$ be the angle between the tangent to Γ_2 and the positive real axis, $s_2(w)$ be the arc length on Γ_2. Let $w = f(z)$ be a homeomorphism of the closure $\overline{G_1}$ of the domain G_1 onto the closure $\overline{G_2}$ of the domain G_2, conformal in open domain G_1.

Theorem 1 ([5]). *If integral moduli of smoothness $\widehat{\omega}_k(\tau_1(s_1),\delta)$ and $\widehat{\omega}_k(\tau_2(s_2),\delta)$ of order k ($k \in \mathbb{N}$) for the functions $\tau_1(s_1)$ and $\tau_2(s_2)$ satisfy Holder condition $\widehat{\omega}_k(\tau_1(s_1),\delta) = O(\delta^\alpha)$ $(\delta \to 0)$ and $\widehat{\omega}_k(\tau_2(s_2),\delta) = O(\delta^\alpha)$ $(\delta \to 0)$, with the same index α, $0 < \alpha < k$, then integral modulus of smoothness $\widehat{\omega}_k(f'(z),\delta)$ of the derivative of the function $f(z)$ on Γ_1 satisfies Holder condition $\widehat{\omega}_k(f'(z),\delta) = O(\delta^\alpha)$ $(\delta \to 0)$ with the same index α.*

Theorem 2 ([6]). *If integral moduli of smoothness $\widehat{\omega}_k(\tau_1(s_1),\delta)$ and $\widehat{\omega}_k(\tau_2(s_2),\delta)$ of order k ($k \in \mathbb{N}$) for the functions $\tau_1(s_1)$ and $\tau_2(s_2)$ satisfy condition $\widehat{\omega}_k(\tau_1(s_1),\delta) = O(\delta^k \log\frac{1}{\delta})$ $(\delta \to 0)$ and $\widehat{\omega}_k(\tau_2(s_2),\delta) = O(\delta^k \log\frac{1}{\delta})$ $(\delta \to 0)$, then integral modulus of smoothness $\widehat{\omega}_k(f'(z),\delta)$ of the derivative of the function $f(z)$ on Γ_1 satisfies condition $\widehat{\omega}_k(f'(z),\delta) = O(\delta^k \log\frac{1}{\delta})$ $(\delta \to 0)$.*

Theorem 3. *Let integral moduli of smoothness $\widehat{\omega}_k(\tau_1(s_1),\delta)$ and $\widehat{\omega}_k(\tau_2(s_2),\delta)$ of order k ($k \in \mathbb{N}$) for the functions $\tau_1(s_1)$ and $\tau_2(s_2)$ satisfy conditions $\widehat{\omega}_k(\tau_1(s_1),\delta) = O(\omega(\delta))$ $(\delta \to 0)$ and $\widehat{\omega}_k(\tau_2(s_2),\delta) = O(\omega(\delta))$ $(\delta \to 0)$, where $\omega(\delta)$ is normal majorant satisfying the condition $\int\limits_0^l \frac{\omega(t)}{t} dt < +\infty$. Then integral* *modulus of smoothness* $\widehat{\omega}_k(f(z),\delta)$ *of the function* $f(z)$ *on* Γ_1 *satisfies the condition* $\widehat{\omega}_k(f(z),\delta) = O(\sigma(\delta))$ $(\delta \to 0)$*, where* $\sigma(\delta)$ *is some integral majorant.*

References

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