Enumeration of topologically non-equivalent functions with one degenerate saddle critical point on triple torus

Oleksandr Kadubovs'kyi
(Donbas State Pedagogical University, Slovians'k, Ukraine)
E-mail: kadubovs@ukr.net

Let $(N, \partial N)$ be a smooth surface with the edge ∂N (∂N can be empty). Let $C^\infty(N)$ denote the space of infinitely differentiable functions on N with edge $\partial N = \partial_- N \cup \partial_+ N$, all critical points of which are isolated and lie in the interior of N, and, furthermore, on the connected components of the edge $\partial_- N$ ($\partial_+ N$) the functions from $C^\infty(N)$ take the same values a (b accordingly).

Two functions f_1 and f_2 from the space $C^\infty(N)$ are called topologically equivalent if there are homeomorphisms $h : N \to N$ and $h' : R^1 \to R^1$ (h' preserves orientation) such that $f_2 = h' \circ f_1 \circ h^{-1}$. If h preserves of the orientation, the functions f_1 and f_2 are called topologically conjugated (eg. [2]) or O-topologically equivalent (eg. [6]).

It is known [2] that a function $f \in C^\infty(N)$, in a certain neighborhood of its isolated critical point $x \in N$ (which is not a local extremum) for which the topological type of level lines changes in passing through x, is reduced by a continuous change of coordinates to the form $f = \mathrm{Re} z^n + c$, $n \geq 2$ (are called «essentially» critical point) or $f = \mathrm{Re} z$ if the topological type of level lines does not change in passing through x. The number of essentially critical points x_i of the function f, together with the values of n_i (exponents in there presentation $f = \mathrm{Re} z^{n_i} + c_i$ in the neighborhoods of the critical points x_i), are called the topological singular type of the function f.

The problem of the topological equivalence of functions from the class $C^\infty(N)$ with the fixed number of critical points was completely solved by V.V. Sharko in [3] and it was established that a finite number of topologically nonequivalent functions of this class exist. However, it should be noted that the task about calculation of topologically non-equivalent functions with the fixed topological singular type is rather complicated and is still **unsolved**.

When considering functions from the class $C(M_g) \subset C^\infty(N)$ that possess **only one** essentially critical point x_0 (degenerate critical point of the saddle type) in addition to local maxima and minima on oriented surface M_g of genus $g \geq 0$, then the problem of counting the number of such non-equivalent functions is greatly simplified. It is well known [2] that $\forall f \in C(M_g)$ the Poincare index of a critical point x_0, is equal $\mathrm{ind}^f(x_0) = 1 - n$, where $n = 2g - 1 + \lambda$ and λ is a total number of local maxima and minima.

Let $C_n(M_g)$ be a class of functions from $C(M_g)$ which, in addition to local maxima and minima, have only one essentially critical point, whose the Poincare index is equal $(1 - n)$. Denote the class of functions from $C(M_g)$ that possess one essentially critical point, k local maxima and l local minima by $C_{k,l}(M_g)$. Then it is clear that $\forall f \in C_{k,l}(M_g)$ $n = 2g - 1 + k + l$.

In the general case, for natural g, k, l (or k, l, and $n = 2g + k + l - 1$), the problem of calculating the number of topologically non-equivalent functions from the class $C_{k,l}(M_g)$ also has proved to be quite a difficult and unsolved problem to date.

The task of calculating the number of topologically non-equivalent functions from the class $C_{1,1}(M_g)$ (for genus $g \geq 1$) was completely solved in [4]. In [5], for natural k and l solved completely the problems of calculating the numbers O-topologically and topologically non-equivalent functions from the class $C_{k,l}(M_0)$ (on two-dimensional sphere).

In [6], [7] solved completely the problems of calculating the numbers O-topologically and topologically non-equivalent functions from the class $C_{1,n-2}(M_1)$ and $C_{1,n-4}(M_2)$ accordingly.

In general case, for functions from the class $C_{1,n-2g}(M_g)$, the task is also still unsolved.

By using the results of [1] we can establish the validity of the following statement

Theorem 1. *For natural $n \geq 7$ the number $d^*(n)$ of O-topologically non-equivalent functions from the class $C_{1,n-6}(M_3)$ can be calculated by the relation*

$$d^*(n) = \frac{1}{n}\left(d(n) + \sum_{j|n,\ j\in\{2;3;4;6;7;8;9;12\}} \phi(j)\cdot\rho\left(n, \tfrac{n}{j}\right)\right), \tag{1}$$

where: $\phi(q)$ is the Euler totient function;

$$d(n) = \frac{1}{72}C_{n+1}^{8}\cdot(9n^4 - 18n^3 - 57n^2 + 34n + 80); \tag{2}$$

$\forall j \in N : \frac{n}{j} \notin N$ *the value* $\rho\left(n, \frac{n}{j}\right) \equiv 0$; $\forall j \in \{2;3;4;6;7;8;9;12\} : \frac{n}{j} \in N$ *the value of* $\rho\left(n, \frac{n}{j}\right)$ *can be calculated by the relations*

$$\begin{aligned} &\rho\left(n,\tfrac{n}{12}\right) = \tfrac{n}{12},\quad \rho\left(n,\tfrac{n}{9}\right) = \tfrac{2n}{9},\quad \rho\left(n,\tfrac{n}{8}\right) = \tfrac{n}{4},\quad \rho\left(n,\tfrac{n}{7}\right) = \tfrac{5n}{7},\quad \rho\left(n,\tfrac{n}{4}\right) = \tfrac{n(n-4)(n+40)}{384},\\ &\rho\left(n,\tfrac{n}{6}\right) = \tfrac{n(n-6)}{72},\quad \rho\left(n,\tfrac{n}{3}\right) = \tfrac{n(n-3)(n-6)(3n+29)}{648},\quad \rho\left(n,\tfrac{n}{2}\right) = \tfrac{n(n-2)(n-4)(n-6)(37n^2+294n-2320)}{46080}. \end{aligned} \tag{3}$$

n	$d(n)$	$d^*(n)$	n	$d(n)$	$d^*(n)$
7	180	30	**19**	1 801 329 010	94 806 790
8	3 044	385	**20**	3 600 529 450	180 028 084
9	26 060	2 900	**21**	6 925 187 830	329 770 930
10	152 900	15 308	**22**	12 869 925 310	584 999 362
11	696 905	63 355	**23**	23 190 544 696	1 008 284 552
12	2 641 925	220 242	**24**	40 637 416 600	1 693 230 295
13	8 691 683	668 591	**25**	69 427 501 000	2 777 100 040
14	25 537 655	1 824 311	**26**	115 901 728 800	4 457 765 752
15	68 396 900	4 559 818	**27**	189 426 912 675	7 015 811 753
16	169 537 940	10 596 558	**28**	303 616 322 295	10 843 450 498
17	393 481 660	23 145 980	**29**	477 960 911 025	16 481 410 725
18	862 928 092	47 941 370	**30**	739 984 318 125	24 666 159 267

TABLE 1. Number $d^*(n)$ of O-topologically non-equivalent functions from the class $C_{1,n-6}(M_3)$

REFERENCES

[1] Goupil A., Schaeffer G. Factoring n-cycles and counting maps of given genus. *European Journal of Combinatorics*, 19(7) : 819–834, 1998.

[2] Prishlyak A.O. Topological equivalence of smooth functions with isolated critical points on a closed surface. *Topologyand its Aplications*, 119(3) : 257–267, 2002.

[3] Sharko V.V. Smooth and Topological Equivalence of Functions on Surfaces. *Ukrainian Mathematical Journal*, 55(5) : 687–700, 2003.

[4] Kadubovs'kyi O.A. Enumeration of topologically non-equivalent smooth minimal functions on a closed surfaces. *Zbirnyk Prats' Institutu Matematyky NAN Ukrainy*, 12(6) : 105–145, 2015.

[5] Kadubovskyi A.A. Enumeration of topologically non-equivalent functions with one degenerate saddle critical pointon two-dimensional sphere, II. *Proceedings of the International Geometry Center*, 8(1) : 46–61, 2015.

[6] Kadubovs'kyi O.A., Balyasa N.P. Enumeration of 2-color chord O-diagrams of the genus one that have one black (or grey) face under rotation and reflection. *Zb. naukovih prac' fiziko-matematicnogo fakul'tetu DDPU*, 6 : 31–46, 2016.

[7] Kadubovs'kyi O.A., Sypchuk Ye. On the number of non-isomorphic 2-color chord O-diagrams of the genus two that have one grey (or black) face. *Zb. naukovih prac' fiziko-matematicnogo fakul'tetu DDPU*, 10 : 21–32, 2020.