A TOPOLOGICAL TRANSFORMATION GROUP OF A HYPERSPACE

Jumaev Davron Ilxomovich
(Tashkent Institute of Architecture and Civil Engineering)
E-mail: d-a-v-ron@mail.ru

Zaitov Adilbek Atakhanovich
(Tashkent Institute of Architecture and Civil Engineering)
E-mail:

Let X be a compact Hausdorff space. By $\exp X$ we denote a set of all nonempty closed subsets of X. A family of sets of the view

$$O\langle U_1, ..., U_n\rangle = \{F \in \exp X : \ F \subset \bigcup_{i=1}^{n} U_n, F \cap U_1 \neq \varnothing, ..., F \cap U_n \neq \varnothing\}$$

forms a base of a topology on $\exp X$, where $U_1, \dots, U_n$ are open nonempty sets in X. This topology is called *the Vietoris topology*. A space $\exp X$ equipped with Vietoris topology is called *hyperspace* of X. For a compact space X its hyperspace $\exp X$ is also a compact space.

Let $f\colon X \to Y$ be continuous map of compacts, $F \in \exp X$. We put

$$(\exp f)(F) = f(F).$$

This equality defines a map $\exp f\colon \exp X \to \exp Y$. For a continuous map f the map $\exp f$ is continuous. Really, it follows from the formula [**?**]

$$(\exp f)^{-1} O\langle U_1, ..., U_m\rangle = O\langle f^{-1}(U_1), ..., f^{-1}(U_m)\rangle$$

what one can check directly. Note that if $f\colon X \to Y$ is an epimorphism, then $\exp f$ is also an epimorphism.

For a Tychonoff space X we put

$$\exp_\beta X = \{F \in \exp \beta X : \ F \subset X\}.$$

It is clear, that $\exp_\beta X \subset \exp \beta X$. Consider the set $\exp_\beta X$ as a subspace of the space $\exp \beta X$. For a Tychonoff spaces X the space $\exp_\beta X$ is also a Tychonoff space with respect to the induced topology.

For a continuous map $f\colon X \to Y$ of Tychonoff spaces we put

$$\exp_\beta f = (\exp \beta f)|_{\exp_\beta X},$$

where $\beta f\colon \beta X \to \beta Y$ is the Stone-Cěch compactification of f (it is unique).

For a Tychonoff space X put

$$\exp(\mathrm{Homeo}\,(X)) = \{\exp(g) : \ g \in \mathrm{Homeo}\,(X)\}.$$

Proposition 1. *For an arbitrary Tychonoff space X we have*

$$\exp(\mathrm{Homeo}\,(X)) \subset \mathrm{Homeo}\,(\exp(X)).$$

Note that the inclusion cannot be reversed.

Example 1. Let $X = \{a, b\}$ be a two-point discrete space. Then $\exp X$ is three point discrete space. There exist only two homeomorphisms of X onto itself: $h, h'\colon X \to X$, defined by the rules $h(a) = a$, $h(b) = b$ and $h'(a) = b$, $h'(b) = a$. At the same time $\exp X$ has six different homeomorphisms four of them could not be generated by h and h'.

For a topological transformation group (G, X, α) we put

$$\exp(G) = \{\exp(\alpha_g) : g \in G\},$$

here $\alpha_g(x) = g(x)$.

Let U_g be an open in G neighbourhood of an element $g \in G$. we define a set $U_{\exp(\alpha_g)} = \{\exp(\alpha_h) : h \in U_g\}$ and put

$$\mathfrak{B}_{\exp(\alpha_g)} = \{U_{\exp(\alpha_g)} : U_g \in \tau_G\},$$

here τ_G is the topology on the space G. It is easy to check that the family $\mathfrak{B}_{\exp(\alpha_g)}$ forms a neighbourhood system of the point $\exp(\alpha_g) \in \exp(G)$.

Theorem 1. *The set* $\exp(G)$ *is a topological group with respect to the operation* $\exp(\alpha_{g_1})\exp(\alpha_{g_2}) = \exp(\alpha_{g_1 g_2})$. *Moreover,* $\exp(\alpha_e)$ *is a unit of the group* $\exp(G)$ *and* $\exp(\alpha_g)^{-1} = \exp(\alpha_{g^{-1}})$, $g \in G$.

Now for α it is possible to define the action $\alpha^{\exp}\colon \exp(G) \times \exp(X) \to \exp(X)$ by the rule

$$\alpha^{\exp}(\exp(\alpha_g), F) = \exp(\alpha_g)(F).$$

Proposition 2. *For the topological transformation groups* (G, X, α)*, the triple* $(\exp(G), \exp(X), \alpha^{\exp})$ *is a topological transformation groups.*

Proposition 3. *If the set* $A \subset X$ *is* G*-invariant, then the set* $\exp(A)$ *is* $\exp(G)$*-invariant.*

Proposition 4. *For a topological transformation group* (G, X, α)*, we have*

$$\ker \alpha^{\exp} = \exp(\ker \alpha).$$

Here $\ker \alpha^{\exp} = \{\exp(\alpha_g) \in \exp(G) : \exp(\alpha_g)(F) = F, \forall F \in \exp(X)\}$, $\exp(\ker \alpha) = \{\exp(\alpha_g) \in \exp(G) : g \in \ker \alpha\}$.

Proposition 4 immediately implies

Corollary 5. *The action* $\alpha^{\exp}$ *is effective if and only if the action* α *is effective.*

Note that for the transitive action α of the group G on the space X, the action $\alpha^{\exp}$ induced from α may not be transitive.

Example 6. Let $X = \{x_1, x_2, x_3\}$ be the discrete topological space (all three points are different). Let

$$G = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$

– the discrete topological group of permutations of the index set $\{1, 2, 3\}$. The action $\alpha\colon G \times X \to X$ of the group G on the space X is defined by the rule $\alpha(g, x_i) = x_{g(i)}$. Then α is a transitive action. Moreover, $\alpha_g(x_i) = x_{g(i)}$. It is clear that $\exp(\alpha_g)(\{x_1, x_2, x_3\}) = \{x_1, x_2, x_3\}$ for each $g \in G$. Thus, for no closed subset $F \subset X$ there is no element $\exp(\alpha_g)$ of the group $\exp(G)$ for which $\exp(\alpha_g)(F) = \Phi$, here $\Phi = \{x_1, x_2, x_3\}$, $F \neq \Phi$. Therefore, the action $\alpha^{\exp}$ is not transitive.

Example 6 shows that the action of the group $\exp(G)$ on the space $\exp(X)$ may not be free, although the action of the group G on the space X is free. But, nevertheless, the following is true.

Proposition 7. *Let* $X = \{x_1, \ldots, x_n\}$ *be a finite discrete space,* G *an arbitrary permutation group (supplied by the discrete topology) of the set* X*. Then, for each free action* α *of the group* G *on the space* X*, the corresponding action* $\alpha^{\exp}$ *of the group* $\exp(G)$ *on the space* $\exp(X)$ *is semi-free. In this case, the only point in the space* $\exp(X)$ *that remains motionless under the action of all elements of* $\exp(G)$ *is the set* $\{x_1, \ldots, x_n\}$.

It is clear that if G is a compact group, then $\exp(G)$ is also a compact group.

Theorem 8. *The action* $\alpha^{\exp}\colon \exp(G)\times\exp(X)\to\exp(X)$ *of the compact group* $\exp(G)$ *on the space* $\exp(X)$ *is a closed map.*

The next statement follows from Theorem 8.

Corollary 9. *If* G *is a compact group and* X *is some* G*-space, then for any closed set* $A\subset\exp(X)$ *the set* $\exp(G)(A)$ *is closed in* $\exp(X)$ *and for compact* A *the set* $\exp(G)(A)$ *is compact.*

Theorem 10. *If* $f\colon X\to Y$ *is an equivariant map of one* G*-space to another, then* $\exp(f)\colon \exp(X)\to\exp(Y)$ *is also an equivariant map of* $\exp(G)$*-spaces.*

The normality of the functor exp and Theorem 10 imply

Corollary 11. *If* $f\colon X\to Y$ *is an equivalence of* G*-spaces* X *and* Y*, then* $\exp(f)\colon \exp X\to\exp Y$ *is an equivalence of* $\exp(G)$*-spaces* $\exp X$ *and* $\exp Y$.

References

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