A generalized Lie-algebraic approach to constructing of integrable fractional dynamical systems

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In the paper [1] a new metrized Lie algebra of fractional integral-differential operators has been introduced and the infinite hierarchy of Lax type Hamiltonian flows on its dual space, which is reduced to the hierarchies of the Lax integrable fractional-differential dynamical systems on coadjoint orbits, has been constructed by use of the Adler-Kostant-Symes Lie-algebraic scheme. In our report we propose a generalization of the described in [1] Lie-algebraic approach to constructing of Lax integrable fractional-differential dynamical systems, which is based on the central extension by the Maurer-Cartan 2-cocycle of the mentioned above operator Lie algebra. By means of this generalized approach we obtain the Lax integrable fractional-differential Kadomtsev-Petviashvily hierarchy, whose quasi-classical approximation leads to the Benney type hydrodynamic systems.

Let us consider the Lie algebra $\mathbb{A}_\alpha := \mathbb{A}_0\{\{D^\alpha, D^{-\alpha}\}\}$ (see [1]), which consists of the fractional integral-differential operators in the forms:

$$\mathrm{a}_\alpha := \sum\nolimits_{j\in\mathbb{Z}_+} a_j D^{\alpha(m_\alpha - j)},$$

where $\mathbb{A}_0 := A\{\{D, D^{-1}\}\}$ is the Lie algebra of integral-differential operators, $A := W_2^\infty(\mathbb{R};\mathbb{C})\cap W_\infty^\infty(\mathbb{R};\mathbb{C})$, $D^\alpha : A \to A$ is a Riemann-Liouville fractional derivative, $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, $\mathrm{Re}\,\alpha \neq 0$, $m_\alpha \in \mathbb{Z}_+$ and $a_j \in \mathbb{A}_0$, $j \in \mathbb{Z}_+$, and possesses the standard commutator $[.,.]$ and invariant with respect to this commutator scalar product:

$$(\mathrm{a}_\alpha, \mathrm{b}_\alpha) := \int_{\mathbb{R}} \mathrm{res}_D\,(\mathrm{res}_{D_\alpha}\,(\mathrm{a}_\alpha \circ \mathrm{b}_\alpha D^{-\alpha}))dx,$$

where res_{D_α} denotes a coefficient at $D^{-\alpha}$ for any fractional integral-differential operator as well as res_D denotes a coefficient at D^{-1} for any integral-differential operator. The Lie algebra $\mathbb{A}_\alpha$ allows the splitting into the direct sum of its two Lie subalgebras $\mathbb{A}_\alpha = \mathbb{A}_{\alpha,+} \oplus \mathbb{A}_{\alpha,-}$, where $\mathbb{A}_{\alpha,+}$ is the Lie subalgebra of the formal power series by the operator D^α.

One parameterizes the Lie algebra $\mathbb{A}_0$ by the variable $y \in \mathbb{S}^1$ and constructs the central extension $\hat{\mathbb{A}}_\alpha := \bar{\mathbb{A}}_\alpha \oplus \mathbb{C}$ of the Lie algebra $\bar{\mathbb{A}}_\alpha := \prod_{y\in\mathbb{S}^1}\mathbb{A}_\alpha$ by the Maurer-Cartan 2-cocycle $\omega_2(.,.)$ on $\bar{\mathbb{A}}_\alpha$ with the commutator:

$$[(\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e)] = ([\mathrm{a}_\alpha, \mathrm{b}_\alpha], \omega_2(\mathrm{a}_\alpha, \mathrm{b}_\alpha)), \qquad (\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e) \in \hat{\mathcal{L}}_\alpha, \tag{1}$$

$$[\mathrm{a}_\alpha, \mathrm{b}_\alpha] = \mathrm{a}_\alpha \circ \mathrm{b}_\alpha - \mathrm{b}_\alpha \circ \mathrm{a}_\alpha, \qquad \omega_2(\mathrm{a}_\alpha, \mathrm{b}_\alpha) := \int_{\mathbb{S}^1} (\mathrm{a}_\alpha, \partial \mathrm{b}_\alpha/\partial y)dy.$$

The invariant with respect to the commutator (1) scalar product on $\hat{\mathbb{A}}_\alpha$ is given by the relationship:

$$((\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e)) = \int_{\mathbb{S}^1} (\mathrm{a}_\alpha, \mathrm{b}_\alpha)dy + ed.$$

The Lie-Poisson bracket, deformed by the space endomorphism $\mathcal{R} = (P_+ - P_-)/2 : \bar{\mathbb{A}}_\alpha \to \bar{\mathbb{A}}_\alpha$, takes the form:

$$\{\gamma, \mu\}_{\mathcal{R}}(\tilde{l}_\alpha) = (\tilde{l}_\alpha, [\mathcal{R}\nabla\gamma(\tilde{l}_\alpha), \nabla\mu(\tilde{l}_\alpha)] + [\nabla\gamma(\tilde{l}_\alpha), \mathcal{R}\nabla\mu(\tilde{l}_\alpha)])+$$

$$+c\omega_2(\mathcal{R}\nabla\gamma(\tilde{l}_\alpha),\nabla\mu(\tilde{l}_\alpha))+c\omega_2(\nabla\gamma(\tilde{l}_\alpha),\mathcal{R}\nabla\mu(\tilde{l}_\alpha)),$$

where $\gamma,\mu\in\mathcal{D}(\bar{\mathbb{A}}^*_\alpha)$ are smooth by Frechet functionals on $\bar{\mathbb{A}}^*_\alpha\simeq\bar{\mathbb{A}}_\alpha$, $\tilde{l}_\alpha\in\bar{\mathbb{A}}^*_\alpha$, $c\in\mathbb{C}$, $P_\pm$ being projectors on $\mathbb{A}_{\alpha,\pm}$, and generate the infinite hierarchy of Lax type Hamiltonian flows:

$$\partial\tilde{l}_\alpha/\partial t_j=[(\nabla\gamma_j(\tilde{l}_\alpha))_+,\tilde{l}_\alpha-c\partial/\partial y],\quad(\tilde{l}_\alpha,c)\in\hat{\mathcal{L}_\alpha}^*,\quad j\in\mathbb{N},\tag{2}$$

by means of the Casimir invariants $\gamma_j\in I(\bar{\mathbb{A}}^*_\alpha)$, $j\in\mathbb{N}$, as Hamiltonians. The Casimir invariants satisfy the relationship:

$$[\tilde{l}_\alpha-c\partial/\partial y,\nabla\gamma_j(\tilde{l}_\alpha)]=0,$$

and can be found in the forms:

$$\gamma_j(\tilde{l}_\alpha)=\int_{y\in\mathbb{S}^1}Tr(\tilde{l}^0_\alpha D^{j\alpha})dy,\quad j\in\mathbb{N}.$$

Here the coefficients of the operator $\tilde{l}^0_\alpha:=D^{j\alpha}+\sum_{k\le j-1}\hat{C}_kD^{k\alpha}$, $k\in\mathbb{Z}$, are such that $d\hat{C}_k/dx=0=d\hat{C}_k/dt_j$, $j\in\mathbb{N}$, and obey the equality:

$$(\tilde{l}_\alpha-c\partial/\partial y)\circ\Phi=\Phi\circ(\tilde{l}^0_\alpha-c\partial/\partial y),\quad\Phi=1+\sum_{r\in\mathbb{N}}\Phi_rD^{-r\alpha}.$$

As an example, one studies the reduction of the hierarchy (2) on the coadjoint orbit related with the element

$$\tilde{l}_\alpha=D^{2\alpha}+D^\alpha\hat{v}+\hat{v}D^\alpha+\hat{u}\in\bar{\mathbb{A}}^*_\alpha,$$

where $\hat{u},\hat{v}\in\bar{\mathbb{A}}_0$, when $c=1$. Looking for the gradients of the Casimir invariants in the forms $\nabla\gamma_j(\tilde{l}_\alpha)=D^{m\alpha}+\sum_{k\le j-1}\hat{a}_{k,j}D^{k\alpha}$, $m\in\mathbb{Z}_+$, $k\in\mathbb{Z}$, one obtains the hierarchy of fractional-differential dynamical systems such as

$$\begin{aligned}&d\hat{u}/dt_1=\hat{u}_y+[\hat{v},\hat{u}], && d\hat{u}/dt_2=\hat{u}_y,\\ &d(\hat{v}+D^\alpha\hat{v}D^{-\alpha})/dt_1=[D^\alpha,\hat{u}-\hat{v}^2]D^{-\alpha}, && d(\hat{v}+D^\alpha\hat{v}D^{-\alpha})/dt_2=\hat{v}_y,\end{aligned}$$

$$\begin{aligned}&d\hat{u}/dt_3=\hat{f}_y+[\hat{f},\hat{u}]\\ &d(\hat{v}+D^\alpha\hat{v}D^{-\alpha})/dt_3=\hat{q}_y+([\hat{q}D^\alpha,\hat{u}]+[D^\alpha\hat{u}+\hat{v}D^\alpha\hat{v},\hat{u}]+[\hat{f},\hat{v}D^\alpha+D^\alpha\hat{v}])D^{-\alpha},\\ &[\hat{q},D^{2\alpha}]=D^\alpha[D^\alpha,\hat{u}-\hat{v}^2],\\ &[\hat{f},D^{2\alpha}]=-(\hat{v}_yD^{2\alpha}+D^\alpha\hat{v}_yD^\alpha+D^{2\alpha}\hat{v}_y)-[\hat{v}D^{2\alpha}+D^{2\alpha}\hat{v},\hat{u}]-[D^\alpha\hat{u},\hat{v}D^\alpha+D^\alpha\hat{v}]-\\ &\qquad-[\hat{u}D^\alpha,D^\alpha\hat{v}]-[\hat{v},D^\alpha\hat{v}^2D^\alpha]-[\hat{q}D^\alpha,\hat{v}D^\alpha+D^\alpha\hat{v}],\\ &\qquad\ldots\ldots\end{aligned}\tag{3}$$

The gradients of corresponding Casimir invariants are written as

$$\begin{aligned}&\nabla\gamma_1(\tilde{l}_\alpha)=D^\alpha+\hat{v}+\sum_{k\le0}\hat{a}_{k,1}D^{k\alpha},\\ &\nabla\gamma_2(\tilde{l}_\alpha)=D^{2\alpha}+(\hat{v}+D^\alpha\hat{v}D^{-\alpha})D^\alpha+\hat{u}+\sum_{k\le0}\hat{a}_{k,2}D^{k\alpha},\\ &\nabla\gamma_3(\tilde{l}_\alpha)=D^{3\alpha}+(\hat{v}+D^\alpha\hat{v}D^{-\alpha}+D^{2\alpha}\hat{v}D^{-2\alpha})D^{2\alpha}+\hat{b}D^\alpha+\hat{f}+\sum_{k\le0}\hat{a}_{k,3}D^{k\alpha},\quad\ldots\ldots,\end{aligned}$$

where $\hat{b}=\hat{u}+D^\alpha\hat{u}D^{-\alpha}+\hat{v}D^\alpha\hat{v}D^{-\alpha}+\hat{q}$. The third system in the hierarchy (3) can be considered as a fractional-differential analog of the Kadomtsev-Petviashvily equation.

References

[1] O.Ye. Hentosh, B.Yu. Kyshakevych, D. Blackmore, A.K. Prykarpatski. New fractional nonlinear integrable Hamiltonian systems. *Applied Mathematics Letters*, 88, February, 41–49, 2019.