On the monoid of cofinite partial isometries of positive integers with a bounded finite noise

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We follow the terminology of [2, 4, 5]. For any positive integer j the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is called the *monoid of cofinite isometries of positive integers with the noise j*. It was introduced in [4].

Any inverse semigroup S admits the *minimum group congruence* $\mathfrak{C}_{\mathbf{mg}}$:

$$a\mathfrak{C}_{\mathbf{mg}}b \quad \text{if and only if} \quad \text{there exists} \quad e\in E(S) \quad \text{such that} \quad ea=eb.$$

Proposition 1. *Let γ and δ be elements of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$. Then $\gamma\mathfrak{C}_{\mathbf{mg}}\delta$ in $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ if and only if $n_{\gamma}^{\mathbf{r}}-n_{\gamma}^{\mathbf{d}}=n_{\delta}^{\mathbf{r}}-n_{\delta}^{\mathbf{d}}$. Moreover, the quotient semigroup $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$ by the map*

$$\pi_{\mathfrak{C}_{\mathbf{mg}}}\colon \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}\to\mathbb{Z}(+), \quad \gamma\mapsto n_{\delta}^{\mathbf{r}}-n_{\delta}^{\mathbf{d}}.$$

Example 2. We put $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}=\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}\sqcup\mathbb{Z}(+)$ and extend the multiplications from $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ and $\mathbb{Z}(+)$ onto $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ in the following way:

$$k\cdot\gamma=\gamma\cdot k=k+(\gamma)\pi_{\mathfrak{C}_{\mathbf{mg}}}\in\mathbb{Z}(+), \qquad \text{for all} \quad k\in\mathbb{Z}(+) \quad \text{and} \quad \gamma\in\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}.$$

Theorem 3. *For any positive integer j every Hausdorff shift-continuous topology τ on $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is discrete.*

Proposition 4. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff semitopological semigroup S. Then $I=S\setminus\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a closed ideal of S.*

Theorem 5. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff topological inverse semigroup S. Then $I=S\setminus\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a topological group.*

Corollary 6. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff topological inverse semigroup S. Then the group $S\setminus\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ contains a dense cyclic subgroup.*

Example 7. Let $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a semigroup defined in Example 2. Put M be an arbitrary subset of $\{2,\ldots,j\}$. We define the topology τ_{lc}^{M} on $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ in the following way:

- (i) all elements of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ are isolated points in $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]},\tau_{\mathsf{lc}}^{M}\right)$;
- (ii) for any $k\in\mathbb{Z}(+)$ the family $\mathscr{B}_{\mathsf{lc}}^{M}(k)=\left\{U_i^M(k)\colon i\in\mathbb{N}\right\}$, where

$$U_i^M(k)=\{k\}\cup\left\{\gamma\in\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M]\colon k\preccurlyeq\gamma \text{ and } n_{\gamma}^{\mathbf{d}}\geqslant i\right\},$$

  is the base of the topology τ_{lc}^{M} at the point $k\in\mathbb{Z}(+)$.

Theorem 8. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff locally compact topological inverse semigroup (S,τ). Then (S,τ) topologically isomorphic to the topological inverse semigroup $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]},\tau_{\mathsf{lc}}^{M}\right)$ for some subset M of $\{2,\ldots,j\}$.*

Corollary 9. *For any positive integer j there exists $(j-1)!+1$ distinct topologically non-isomorphic Hausdorff locally compact semigroup inverse topologies on the monoid $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$.*

The obtained results generalize the corresponding results of the papers [1] and [3].

References

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