On homotopy nilpotency of Moore space

**Marek Golasiński**
(University of Warmia and Mazury, Olsztyn, Poland)
*E-mail:* `marekg@matman.uwm.edu.pl`

Given based spaces $X_1, X_2$, we use the customary notations $X_1 \times X_2$ for their Cartesian product, $X_1 \vee X_2$ for their wedge and $X_1 \wedge X_2$ for the smash product of $X_1, X_2$.

Recall that an *H-space* is a pair $(X, \mu)$, where $X$ is a space and $\mu : X \times X \to X$ is a map such that the diagram

$$\begin{tikzcd} X \times X \arrow[r, "\mu"] & X \\ X \vee X \arrow[u, hook] \arrow[ur, "\nabla"'] & \end{tikzcd}$$

commutes up to homotopy, where $\nabla : X \vee X \to X$ is the folding map. An $H$-space $X$ is called a *group-like space* if $X$ satisfies all the axioms of groups up to homotopy. From now on, we assume that any $H$-space $X$ is group-like. For an $H$-space $X$, we write $\varphi_{X,1} = \iota_X$, $\varphi_{X,2} : X \times X \to X$ for the basic commutator map and $\varphi_{X,n+1} = \varphi_{X,2} \circ (\varphi_{X,n} \times \iota_X)$ for $n \geq 2$.

The *nilpotency class* $\operatorname{nil}(X, \mu)$ of an $H$-space $(X, \mu)$ is the least integer $n \geq 0$ for which the map $\varphi_{X,n+1} \simeq *$ is nullhomotopic and we call the homotopy associative $H$-space $X$ homotopy nilpotent. If no such integer exists, we put $\operatorname{nil}(X, \mu) = \infty$. In the sequel, we simply write $\operatorname{nil} X$ for the nilpotency class of an $H$-space $X$.

In virtue of [2, 2.7. Theorem], we have

**Theorem 1.** *If $X$ is an $H$-space then*

$$\operatorname{nil} X = \sup_m \operatorname{nil}[X^m, X] = \sup_m \operatorname{nil}[X^{\wedge m}, X] = \sup_Y \operatorname{nil}[Y, X],$$

*where $m$ ranges over all integers and $Y$ over all topological spaces.*

Then, by means of [8, Lemma 2.6.1], we may state

**Corollary 2.** *A connected $H$-space $X$ is homotopy nilpotent if and only if the functor $[-, X]$ on the category of all spaces is nilpotent group valued.*

With any based space $X$, we associate the integer $\operatorname{nil} \Omega(X)$ called the *nilpotency class* of $X$ for the loop space $\Omega(X)$ on $X$. Although many results on the homotopy nilpotency have been obtained, the homotopy nilpotency classes have been determined in very few cases.

**Example 3.** (1) It is well-known that

$$\operatorname{nil} \Omega(\mathbb{S}^n) = \begin{cases} 3 & \text{for } n \text{ even with } n \neq 2; \\ 2 & \text{for } n \text{ odd with } n \neq 1, 3, 7 \text{ or } n = 2; \\ 1 & \text{for } n = 1, 3, 7 \end{cases}$$

for the $n$-sphere $\mathbb{S}^n$.

(2) For the wedge $\mathbb{S}^m \vee \mathbb{S}^n$ of two spheres with $m, n \geq 2$, we have

$$\operatorname{nil} \Omega(\mathbb{S}^m \vee \mathbb{S}^n) = \infty.$$

Write $\mathbb{K}P^m$ for the projective $m$-space for $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions. Then, results from [6] have been applied in [3] to study extensively the homotopy nilpotency of the loop spaces of Grassmann and Stiefel manifolds over $\mathbb{K}$, and their $p$-localization.

Let $\mathbb{S}^{2m-1}_{(p)}$ be the $p$-localization of the sphere $\mathbb{S}^{2m-1}$ at a prime $p$. The main result of the paper [4] is the explicit determination of the homotopy nilpotence class of a wide range of homotopy associative multiplications on localized spheres $\mathbb{S}^{2m-1}_{(p)}$ for $p>3$.

Next, let $A$ be an Abelian group and $n$ any integer $\geq 2$. A $CW$-complex $X$ satisfying $\pi_j(X)=0$ for $j<n$, $\pi_n(X)\approx A$ and $H_i(X)=0$ for $i>n$ is known as a *Moore space* of type $(A,n)$, or simply an $M(A,n)$ space. By [7], it is known that a Moore space $M(A,n)$ with $n\geq 2$ exists and, in view of [5, Example 4.34], the homotopy type of a Moore space $M(A,n)$ is uniquely determined by $A$ and $n\geq 2$. This implies that every Moore space $M(A,n)$ with $n\geq 3$, is the suspension $\Sigma M(A,n-1)$. Furthermore, in [1, Section 2], it was shown that also $M(A,2)$ is the suspension $\Sigma L(A)$ for some $CW$-complex $L(A)$.

Now, we examine the homotopy nilpotency of $M(A,n)$ with $\geq 2$. Notice that $\mathbb{S}^n=M(\mathbb{Z},n)$ and the wedge $\mathbb{S}^n\vee\mathbb{S}^n=M(\mathbb{Z}\oplus\mathbb{Z},n)$ for the integers $\mathbb{Z}$. Then, by Example 3, we have that $\operatorname{nil}\Omega(\mathbb{S}^n)\leq 3$ but $\operatorname{nil}\Omega(\mathbb{S}^n\vee\mathbb{S}^n)=\infty$ for $n\geq 2$.

First, we show the general fact

**Proposition 4.** *If the reduced homology $\tilde{H}_*(X,\mathbb{F})$ has at least two primitive generators, where $\mathbb{F}$ is a field then $\Omega\Sigma(X)$ is not homotopy nilpotent.*

Then, we state the main resut

**Theorem 5.** *Let $m\geq 1$, $n_1,\ldots,n_m\geq 2$ and $M(A_k,n_k)$ be Moore spaces of type $(A_k,n_k)$ for $k=1,\ldots,m$. Then:*

(1) $\operatorname{nil}\Omega((M(A_1,n_1)\times\cdots\times(M(A_m,n_m))<\infty$ *if and only if if $A_k$ are torsion-free groups with rank* $\mathrm{r}(A_k)=1$ *for $k=1,\ldots,m$;*

(2) $\operatorname{nil}\Omega((M(A_1,n_1)\vee\cdots\vee M(A_m,n_m))<\infty$ *if and only if $m=1$ and $A_1$ is a torsion-free group with rank* $\mathrm{r}(A_1)=1$.

In particular, we derive

**Corollary 6.** *If $M(A,n)$ is a Moore space with $n\geq 2$ then*

$$\operatorname{nil}\Omega(M(A,n))<\infty$$

*if and only if $A$ is a torsion-free group with rank $r(A)=1$ or equivalently, $A$ is a subgroup of the rationals $\mathbb{Q}$.*

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