Geometry of curves in three-dimensional space and invariants of nonlinear differential equations of the second order

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The talk is devoted to differential invariants of nonlinear differential equations of the second order of the form

$$u_{tt} = [\alpha(u)]_{xx} + [\beta(u)]_x + \gamma(u). \tag{1}$$

where t, x are independent variables and α, β, γ are smooth functions of u.

We considered admissible point transformations only, i.e. transformations of the space of 0-jets $J^0(\mathbb{R}^2)$ that preserve the class of such equations (see, for example, [1, 2]). Admissible transformations form a six-dimensional Lie group with Lie algebra $\mathcal{G}$ thai is generated by the admissible vector fields

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial u} \quad t\frac{\partial}{\partial t}, \quad x\frac{\partial}{\partial x}, \quad u\frac{\partial}{\partial u}. \tag{2}$$

The first three vector fields correspond to translation along the x, y, u axes, and the last three ones correspond to homothety.

Write equation (1) in the following form:

$$u_{tt} = a'(u)u_x^2 - a(u)u_{xx} - b(u)u_x - c(u),$$

where

$$a(u) = \alpha'(u), \quad b(u) = \beta'(u), \quad c(u) = \gamma(u).$$

Consider the following one-dimensional trivial bundle

$$\pi : \mathbb{R}^3 \to \mathbb{R}, \quad \pi : (a, b, c) \mapsto u.$$

A section of this bundle are parametric curve in $\mathbb{R}^3$ that correspond to equation (1). Let $J^k(\pi)$ be the space of k-jets of sections of π with canonical coordinates $u, a_0, b_0, c_0, \ldots, a_k, b_k, c_k$.

Restriction of the Lie algebra of admissible vector fields to the space $J^0(\pi)$ is given by the following vector fields:

$$\frac{\partial}{\partial u}, \quad 2a_0\frac{\partial}{\partial a_0} + b_0\frac{\partial}{\partial b_0}, \quad a_0\frac{\partial}{\partial a_0} + b_0\frac{\partial}{\partial b_0} + c_0\frac{\partial}{\partial c_0}, \quad u\frac{\partial}{\partial u} - c_0\frac{\partial}{\partial c_0}$$

Theorem 1. *The algebra of differential invariants of equations (1) is generated by the following functions:*

$$I_{a,k} = \frac{a_k b_0^{2k}}{a_0^{k+1} c_0^k}, \quad I_{b,k} = \frac{b_k b_0^{2k-1}}{a_0^k c_0^k}, \quad I_{c,k} = \frac{c_k b_0^{2k}}{a_0^k c_0^{k+1}},$$

where $k = 1, 2, \ldots$.

The constructed invariants are analogs of curvature and torsion for curves in three-dimensional Euclidean space.

References

[1] Kushner A. G., Lychagin V. V. Petrov invariants for 1-d control hamiltonian systems // Global and Stochastic Analysis. – 2012. – Vol. 2, no. 1. – P. 241-264.

[2] Kushner E.N., Kushner A.G., Samohin A.V. Differential invariants of third order evolutionary non-linear PDEs // 2019 Twelfth International Conference Management of large-scale system development (MLSD). IEEE Piscataway, NJ, United States, 2019.