Deformations of circle-valued Morse functions on 2-torus

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Let M be a smooth compact surface, X be a closed (possible empty) subset of M. By P we also denote either $\mathbb{R}$ or S^1. The group $\mathcal{D}(M, X)$ of diffeomorphisms of M fixed on X acts from the right on the space of smooth maps $C^\infty(M, P)$ by the rule

$$\gamma : C^\infty(M,P) \times \mathcal{D}(M,X) \to C^\infty(M,P), \qquad \gamma(f,h) = f \circ h.$$

With respect to γ we denote by

$$\mathcal{S}(f,X) = \{h \in \mathcal{D}(M,X) \,|\, f \circ h = f\},$$
$$\mathcal{O}(f,X) = \{f \circ h \,| h \in \mathcal{D}(M,X)\}$$

the *stabilizer* and the *orbit* of $f \in C^\infty(M,P)$. Endow strong Whitney C^∞-topologies on $C^\infty(M,P)$ and $\mathcal{D}(M,X)$; then for a map $f \in C^\infty(M,P)$ these topologies induce some topologies on $\mathcal{S}(f,X)$ and $\mathcal{O}(f,X)$. We denote by $\mathcal{D}_{\mathrm{id}}(M,X)$ a connected component of the identity map $\mathcal{D}(M,X)$, and by $\mathcal{O}_f(f,X)$ a connected component of $\mathcal{O}(f,X)$ containing f. If $X = \varnothing$ we omit the symbol "$\varnothing$" from our notation.

To state our main result we need a notion of wreath product of groups of a special kind. Let G be a group, $n \geq 1$ be an integer. A semi-direct product $G^n \rtimes \mathbb{Z}$ with respect to a non-effective $\mathbb{Z}$-action α on G^n by cyclic shifts

$$\alpha(b_0, b_1, \dots, b_{n-1}; k) = (b_k, b_{1+k}, \dots, b_{n+k-1}),$$

where all indexes are taken modulo n, will be denoted by $G \wr_n \mathbb{Z}$ and called a *wreath product* of G with $\mathbb{Z}$ under n.

The following theorem is our main result.

Theorem 1 ([1]). *Let f be a function from $\mathcal{F}(T^2, P)$ with at least one critical point and whose Kronrod-Reeb graph contains a cycle. Then there exist a cylinder $Q \subset T^2$ such that $f|_Q : Q \to P$ is a Morse function, $n \in \mathbb{N}$ such that there is an isomorphism*

$$\pi_1 \mathcal{O}_f(f) \cong \pi_0 \mathcal{S}'(f|_Q, \partial Q) \wr_n \mathbb{Z},$$

where $\mathcal{S}'(f|_Q, \partial Q) = \mathcal{S}(f|_Q, \partial Q) \cap \mathcal{D}_{\mathrm{id}}(Q, \partial Q)$.

References

[1] Bohdan Feshchenko. Deformations of circle-valued Morse functions on 2-torus. Submitted to Proceedings of the International Geometry Center, arXiv2104.06151