On the properties smooth manifolds defined by intersections

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The report is devoted to study properties and construction an examples of the $(3,4)$-dim smooth manifolds contained the surfaces of constant curvature.

At the first will be considered the Dyck-surface (W.Dyck,1888) defined by the algebraic equation

$$\left({z_1}^2+{z_2}^2\right)\left({z_1}^2+{z_2}^2+{z_3}^2\right)-z_3\left(4\,{z_1}^2+2\,{z_2}^2\right)=0, \tag{1}$$

where z_1, z_2, z_3 are the complex coordinates: $z_1 = x + Ia$, $z_2 = y + Ib$, $z_3 = z + Ic$, $I^2 = -1$. The complex surface (1) is generalization of real projective surface and belongs to the class of the one-side surfaces having an important applications in various branch of modern algebraic topology (J.Milnor,1968).

Proposition 1. *Joint consideration both equations* (1) *and the equation of the* $5D$*-sphere* $|z_1|^2+|z_2|^2+|z_3|^2=1$ *with real coordinates* (x,y,z,a,b,c)*, in general, lead us to the some* $4D$*-space, where* (t)*-is auxiliary parameter:*

$${}^4G(a,b,c,z,t)=bc^2-b+b^3+bz^2+2\,ct-ztc-bt^2+ba^2=0, \tag{2}$$

containing $3D$*-subspace with the equation*

$${}^3H(a,b,c,z)=4\,a^2b^2-4\,b^2+4\,b^4+4\,c^2+4\,b^2c^2-4\,zc^2+4\,b^2z^2+z^2c^2=0. \tag{3}$$

With the equation (3) *at the condition* $b=b(a)$ *can be associated an invariant second order ODE of the form* $b''-A_1b'^3-3A_2b'^2-3A_3b'-A_4=0$, $A_i=A_i(a,b)$ *having the General integral*

$${}^3F(a,b,C_1,C_2)=-\,C_1-\,b^4-\,a^2b^2+2\,b^2C_2=0.$$

having an algebraic curve of genus $g_{a,b}=1$*, and which is placed on the* $2D$*-surface, equipped by the metrics of const positive curvature* $K=1$

$$\phi^2ds^2=\psi_1(x,y)dx^2+2\psi_2(x,y)dx\,dy+\psi_3(x,y)dy^2,\quad \phi(x,y)=\psi_1(x,y)\psi_3(x,y)-\left(\psi_2(x,y)\right)^2. \tag{4}$$

The components ψ_i *of the metrics are determined from the system (M.R.Liouville,1897)*

$$\psi_{1x}+2\,A_3\psi_1-2\,A_4\psi_2=0,\quad \psi_{3y}+2\,A_1\psi_2-2\,A_2\psi_3=0,$$

$$\psi_{1y}+2\psi_{2x}-2\,A_3\psi_2+4\,A_2\psi_1-2\,A_4\psi_3=0,\quad \psi_{3x}+2\,\psi_{2y}+2\,A_2\psi_2-4\,A_3\psi_3+2\,A_1\psi_1=0. \tag{5}$$

In the second part of report we consider some examples of the Brieskorn type manifolds which are the intersection of the fife-dimensional sphere with the singular manifold ($l=2,\quad m=3,\quad n=5$)

$$|z_1|^2+|z_2|^2+|z_3|^2=1,\quad z_1^2+z_2^3+z_3^5=0. \tag{6}$$

Proposition 2. *From the equations of the system* (6)

$$x^2+a^2+y^2+b^2+z^2+c^2-1=0,\quad 2\,xa+3\,y^2b-b^3+5\,z^4c-10\,z^2c^3+c^5=0,$$

$$x^2-a^2+y^3-3\,yb^2+z^5-10\,z^3c^2+5\,zc^4=0, \tag{7}$$

on the six real coordinates $z_1=x+Ia$, $z_2=y+Ib, z_3=z+Ic$*, in the case the relation* $z=z(y)$ *holds, the linearizable the second order ODE*

$$\frac{d^2}{dy^2}z(y)=\frac{\left(\frac{d}{dy}z(y)\right)\left(-y\frac{d}{dy}z(y)+z(y)\right)}{yz(y)}$$

can be derived. It has general integral

$$F(z,y,C_1,C_2)=z-\sqrt{C_1\,y^2+2\,C_2}=0,$$

containing the $2D$*-surface of constant curvature with the components of the metrics defined from* (5),

$$\psi_2(x,y)=\frac{\left(C_3\,x^2+C_4\right)y^{5/3}}{\sqrt[3]{x}}+\frac{C_1\,x^2+C_2}{\sqrt[3]{x}\sqrt[3]{y}},\ \psi_1(x,y)=-\frac{x^{2/3}\left(-C_6+C_3\,y^4+2\,C_1\,y^2\right)}{y^{4/3}},$$

$$\psi_3(x,y)=-\frac{y^{2/3}\left(-C_5+x^4C_3+2\,x^2C_4\right)}{x^{4/3}}$$

with the parameters C_i.

By analogy are considered the case of tetrahedral space which corresponds to the intersection of the fife-dimensional sphere with singular manifold ($l=2,\quad m=3,\quad n=4$*)*

$$|z_1|^2+|z_2|^2+|z_3|^2=1,\quad z_1^2+z_2^3+z_3^4=0, \tag{8}$$

for which corresponding ODE has the form

$$\frac{d^2}{dy^2}z(y)=1/2\,\frac{\left(\frac{d}{dy}z(y)\right)\left(3\,\left(\frac{d}{dy}z(y)\right)^2y^2-6\,\left(\frac{d}{dy}z(y)\right)yz(y)+2\,(z(y))^2\right)}{y\,(z(y))^2}$$

and the octahedral space defined by the condition

$$|z_1|^2+|z_2|^2+|z_3|^2=1,\quad z_1^2+z_2^3+z_2z_3^3=0, \tag{9}$$

with the ODE: $\frac{d^2}{dy^2}z(y)=-\frac{\left(\frac{d}{dy}z(y)\right)^2}{z(y)}$, *and corresponding metrics* (4) *with the components* $\psi_i(x,y)$

$$\psi_3(x,y)=-2\,y^{2/3}C_3\,x^2-4\,y^{2/3}C_4\,x+C_5\,y^{2/3},\ \psi_2(x,y)=\frac{y^2\,C_3\,x+y^2\,C_4+C_1\,x+C_2}{\sqrt[3]{y}},$$

$$\psi_1(x,y)=\frac{-1/2\,C_3\,y^4-C_1\,y^2+C_6}{y^{4/3}}.$$

To studying a moore detail properties of considered spaces can be used the $4D$-Riemann extensions

$$ds^2=2\,(zA_3-tA_4)\,dx^2+4\,(zA_2-tA_3)\,dx\,dy+2\,(zA_1-tA_2)\,dy^2+2\,dy\,dz+2\,dy\,dt$$

of $2D$-metrics and with help of the Liouville-Tresse-Cartan invariants to investigated topological properties of the Brieskorn manifolds.

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