Problem on extremal decomposition of the complex plane

Aleksandr Bakhtin, Iryna Denega
(Institute of mathematics of the National Academy of Sciences of Ukraine, Complex analysis and potential theory Department, 3 Tereschenkivska St, Kyiv, Ukraine, 01024)
E-mail: abahtin@imath.kiev.ua, iradenega@gmail.com

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, let $\mathbb{C}$ be the complex plane, and let $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be its one-point compactification, $\mathbb{R}^+ = (0, \infty)$. Let $r(B, a)$ be the inner radius of the domain $B \subset \overline{\mathbb{C}}$ relative to a point $a \in B$. The inner radius of the domain B is connected with Green's generalized function $g_B(z, a)$ of the domain B by the relations

$$g_B(z, a) = -\ln|z - a| + \ln r(B, a) + o(1), \quad z \to a,$$

$$g_B(z, \infty) = \ln|z| + \ln r(B, \infty) + o(1), \quad z \to \infty.$$

Definition 1. Let $n \in \mathbb{N}$, $n \geqslant 2$. The system of points $A_n := \left\{a_k \in \mathbb{C} : \, k = \overline{1, n}\right\}$ is called n-ray, if $|a_k| \in \mathbb{R}^+$ for $k = \overline{1, n}$ and $0 = \arg a_1 < \arg a_2 < \ldots < \arg a_n < 2\pi$.

Denote $\alpha_k := \dfrac{1}{\pi} \arg \dfrac{a_{k+1}}{a_k}$, $\alpha_{n+1} := \alpha_1$, $k = \overline{1, n}$, $\sum\limits_{k=1}^{n} \alpha_k = 2$.

Problem 2. (V.N. Dubinin [1, 2]) For all values of the parameter $\gamma \in (0, n]$ to show that the maximum of the functional

$$I_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where B_0, B_1, B_2,..., B_n, $n \geqslant 2$, are pairwise disjoint domains in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1, n}$, is attained for the configuration of domains B_k and points a_k which possesses the n-fold symmetry.

In work [1], the above-formulated problem was solved for the value of the parameter $\gamma = 1$ and all values of the natural parameter $n \geqslant 2$. Namely, it was shown that the following inequality holds

$$r(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant r\left(D_0, 0\right) \prod_{k=1}^{n} r\left(D_k, d_k\right),$$

where d_k, D_k, $k = \overline{0, n}$, are the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2 - 1)w^n + 1}{w^2(w^n - 1)^2}\, dw^2.$$

In work [3], L.V. Kovalev got its solution for definite sufficiently strict limitations on the geometry of arrangement of the systems of points on a unit circle, namely, for systems of points for which the following inequalities hold

$$0 < \alpha_k \leqslant 2/\sqrt{\gamma}, \quad k = \overline{1, n}, \quad n \geqslant 5.$$

In work [4], it was shown that the result by L.V. Kovalev is true for $n = 4$. The solution of this problem for $\gamma \in (0, 1]$ was given in work [5]. Some partial cases of this problem were studied, for example, in [6–10].

For the further analysis, we calculate the quantity

$$I_n^0(\gamma) = r^{\gamma}\left(D_0, 0\right) \prod_{k=1}^{n} r\left(D_k, d_k\right),$$

where d_k, D_k, $k=\overline{0,n}$, $d_0=0$, are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2=-\frac{(n^2-\gamma)w^n+\gamma}{w^2(w^n-1)^2}\,dw^2.$$

As was shown in [1, 2, 3, 6], the quantity $I_n^0(\gamma)$ takes the form

$$I_n^0(\gamma)=\left(\frac{4}{n}\right)^n\frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}}\left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}.$$

Theorem 3. [9] *Let $\gamma\in(1,2]$. Then, for any different points a_1 and a_2 of a unit circle and any mutually disjoint domains B_0, B_1, B_2, $a_1\in B_1\subset\overline{\mathbb{C}}$, $a_2\in B_2\subset\overline{\mathbb{C}}$, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, the inequality*

$$r^\gamma\left(B_0,0\right)r\left(B_1,a_1\right)r\left(B_2,a_2\right)\leqslant I_2^0(\gamma)\left(\frac{1}{2}\left|a_1-a_2\right|\right)^{2-\gamma}.$$

is true. The sign of equality in this inequality is attained, when the points a_0, a_1, a_2 and the domains B_0, B_1, B_2 are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2=-\frac{(4-\gamma)w^2+\gamma}{w^2(w^2-1)^2}dw^2.$$

Remark 4. Theorem 3 yields the complete solution of the above-posed problem of finding the maximum of product of inner radii of two domains relative to the points of a unit circle on the degree γ of the inner radius of the domain relative to the origin at arbitrary $\gamma\in(0,2\,]$, provided that all three domains are mutually non-overlapping domains.

Theorem 5. [9] *Let $n\in\mathbb{N}$, $n\geqslant 3$, $\gamma\in(1,n]$. Then, for any system of different points $A_n=\{a_k\}_{k=1}^n\in\mathbb{C}\backslash\{0\}$ of a unit circle and for any collection of mutually disjoint domains B_0, B_k, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{1,n}$, the following inequality holds*

$$r^\gamma(B_0,0)\prod_{k=1}^n r(B_k,a_k)\leqslant\left(\sin\frac{\pi}{n}\right)^{n-\gamma}\left(I_2^0\left(\frac{2\gamma}{n}\right)\right)^{\frac{n}{2}}.$$

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