Asymptotically equivalent subspaces of metric spaces

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We investigate the asymptotic behavior of unbounded metric spaces at infinity. To do this we consider a sequence of rescaling metric spaces $\left(X, \frac{1}{r_n}d\right)$ generated by a metric space (X,d) and a scaling sequence $(r_n)_{n\in\mathbb{N}}$ of positive reals with $r_n \to \infty$. By definition, the pretangent spaces to (X,d) at infinity $\Omega^X_{\infty,\tilde{r}}$ are limit points of this rescaling sequence. We found the necessary and sufficient conditions under which two given unbounded subspaces of (X,d) have the same pretangent spaces at infinity.

Definition 1. Let (X,d) be an unbounded metric space. Two sequences $\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset X$ and $\tilde{y} = (y_n)_{n\in\mathbb{N}} \subset X$ are *mutually stable* with respect to a scaling sequence $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ if there is a finite limit

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n}.$$

For every unbounded metric space (X,d) and every scaling sequence $\tilde{r}$, we denote by $Seq(X,\tilde{r})$ the set of all sequences $\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset X$ for which $\lim\limits_{n\to\infty} d(x_n,p) = \infty$ and there is a finite limit

$$\lim_{n\to\infty} \frac{d(x_n, p)}{r_n},$$

where p is a fixed point of X.

Definition 2. A set $F \subseteq Seq(X,\tilde{r})$ is *self-stable* if any two $\tilde{x}, \tilde{y} \in F$ are mutually stable. F is *maximal self-stable* if it is self-stable and, for arbitrary $\tilde{y} \in Seq(X,\tilde{r})$, we have either $\tilde{y} \in F$ or there is $\tilde{x} \in F$ such that $\tilde{x}$ and $\tilde{y}$ are not mutually stable.

Let (X,d) be an unbounded metric space, let Y and Z be unbounded subspaces of X and let $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ be a scaling sequence.

Definition 3. The subspaces Y and Z are *asymptotically equivalent* with respect to $\tilde{r}$ if for every

$$\tilde{y}_1 = (y_n^{(1)})_{n\in\mathbb{N}} \in Seq(Y,\tilde{r}) \quad \text{and} \quad \tilde{z}_1 = (z_n^{(1)})_{n\in\mathbb{N}} \in Seq(Z,\tilde{r})$$

there exist

$$\tilde{y}_2 = (y_n^{(2)})_{n\in\mathbb{N}} \in Seq(Y,\tilde{r}) \quad \text{and} \quad \tilde{z}_2 = (z_n^{(2)})_{n\in\mathbb{N}} \in Seq(Z,\tilde{r})$$

such that

$$\lim_{n\to\infty} \frac{d(y_n^{(1)}, z_n^{(2)})}{r_n} = \lim_{n\to\infty} \frac{d(y_n^{(2)}, z_n^{(1)})}{r_n} = 0.$$

We shall say that Y and Z are *strongly asymptotically equivalent* if Y and Z are asymptotically equivalent for all scaling sequences $\tilde{r}$.

Let (X,d) be a metric space and let $p \in X$. For every $t > 0$ we denote by $S(p,t)$ the sphere with the radius t and the center p,

$$S(p,t) := \{x \in X\colon d(x,p) = t\},$$

and for every $Y \subseteq X$ we write

$$S_t^Y := S(p,t) \cap Y.$$

Let Y and Z be subspaces of (X,d). Define

$$\varepsilon(t,Z,Y) := \sup_{z \in S_t^Z} \inf_{y \in Y} d(z,y)$$

and

$$\varepsilon(t) = \max\{\varepsilon(t,Z,Y), \varepsilon(t,Y,Z)\},$$

where we set $\varepsilon(t,Z,Y) = 0$ if $S_t^Z = \varnothing$ and, respectively, $\varepsilon(t,Y,Z) = 0$ if $S_t^Y = \varnothing$.

Theorem 4. *Let Y and Z be unbounded subspaces of a metric space (X,d). Then Y and Z are strongly asymptotically equivalent if and only if*

$$\lim_{t\to\infty} \frac{\varepsilon(t)}{t} = 0.$$

Corollary 5. *Let (X,d) be an unbounded metric space and let Y be an unbounded subspace of X. Then the following conditions are equivalent.*

(1) *For every $\tilde{r}$ and every maximal self-stable $\tilde{X}_{\infty,\tilde{r}} \subseteq Seq(X,\tilde{r})$ there is a maximal self-stable $\tilde{Y}_{\infty,\tilde{r}} \subseteq Seq(X,\tilde{r})$ such that $\tilde{Y}_{\infty,\tilde{r}} \subseteq \tilde{X}_{\infty,\tilde{r}}$ and the embedding $E_{m_Y} : \Omega^Y_{\infty,\tilde{r}} \to \Omega^X_{\infty,\tilde{r}}$ is an isometry.*

(2) *The equality*

$$\lim_{t\to\infty} \frac{\varepsilon(t,X,Y)}{t} = 0$$

holds.

(3) *X and Y are strongly asymptotically equivalent.*

Remark 6. Theorem 4 and Corollary 5 can be considered as asymptotic variants of previously proved facts from [1].

References

[1] Oleksiy Dovgoshey. Tangent spaces to metric spaces and to their subspaces. *Ukr. Mat. Visn.*, 5: 470–487, 2008; Reprinted in *Ukr. Mat. Bull.*, 5(4): 457–477, 2008.